IMPLEMENTING ARROW-DEBREU EQUILIBRIA BY TRADING INFINITELY-LIVED SECURITIES

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Abstract

We show that Arrow-Debreu equilibria with countably additive prices in infinite-time economy under uncertainty can be implemented by trading infinitely-lived securities in complete sequential markets under two different portfolio feasibility constraints: wealth constraint, and essentially bounded portfolios. Sequential equilibria with no price bubbles implement Arrow-Debreu equilibria, while those with price bubbles implement Arrow-Debreu equilibria with transfers. Transfers are equal to price bubbles on initial portfolio holdings. Price bubbles arise in sequential equilibrium under the wealth constraint if some securities are in zero supply or negative prices are permitted, but cannot arise with essentially bounded portfolios.

1. Introduction

Equilibrium models of dynamic competitive economies extending over infinite time play an important role in contemporary economic theory. The basic solution concept for such models is the Arrow-Debreu (or Walrasian) equilibrium. In Arrow-Debreu equilibrium it is assumed that agents simultaneously trade arbitrary consumption plans for the entire infinite and state-contingent future. In applied work, on the other hand, a different market structure and equilibrium concept are used: instead of trading arbitrary consumption plans at a single date, agents trade securities in sequential markets at every date in every event. The importance of Arrow-Debreu equilibrium rests on the possibility of *implementing* equilibrium allocations by trading suitable securities in sequential markets.

The idea of implementing an Arrow-Debreu equilibrium allocation by trading securities takes its origin in the classical paper by Arrow [3]. Arrow proved that every Arrow-Debreu equilibrium allocation in a two-date economy can be implemented by trading in complete security markets at the first date and spot commodity markets at every date in every event. The implementation is *exact*—the sets of equilibrium allocations in the two market structures are exactly the same. Arrow's result can be easily extended to a multidate economy with finite time-horizon. Duffie and Huang [7] proved that Arrow-Debreu equilibria can be implemented by trading securities in continuous-time finite-horizon economy.

In this paper we study implementation of Arrow-Debreu equilibrium allocations by sequential trading of infinitely-lived securities in an infinite-time economy. Our results extend those of Kandori [11], from the setting of a representative consumer, and our previous results (Huang and Werner [10]), from the setting of no uncertainty and a single security, to the general setting of multiple consumers, multiple securities, and uncertainty. Wright [20] studied implementation in infinite-time economies with one-period-lived securities.

The crucial aspect of implementation in infinite-time security markets is the choice of feasibility constraints on agents' portfolio strategies. A feasibility con-

straint has to be imposed for otherwise agents would be able to borrow in security markets and roll over the debt without ever repaying it (Ponzi scheme). However, the constraint cannot be too "tight" for it could prevent agents from using portfolio strategies that generate wealth transfers necessary to achieve consumption plans of an Arrow-Debreu equilibrium. Wright [20] employs the *wealth constraint* which says that a consumer cannot borrow more than the present value of her future endowments. He proved that exact implementation holds with one-period-lived securities—the set of Arrow-Debreu equilibrium allocations and the set of equilibrium allocations in complete sequential markets are the same.

The difficulty in extending implementation results to infinitely-lived securities lies in the possibility of price bubbles in sequential markets. Kocherlakota [12], Magill and Quinzii [14], and Huang and Werner [10] pointed out that the wealth constraint gives rise to sequential equilibria with price bubbles on securities that are in zero supply. We demonstrate in this paper that, if one does not exclude negative security prices, then there exist sequential equilibria with price bubbles under the wealth constraint even if the supply of securities is strictly positive. We prove that Arrow-Debreu equilibria with countably additive prices can be implemented by trading infinitely-lived securities in complete sequential markets under the wealth constraint with no price bubbles. That is, the set of Arrow-Debreu equilibrium allocations is the same as the set of equilibrium allocations in sequential markets with no price bubbles. Further, we show that sequential equilibria with nonzero price bubbles correspond to Arrow-Debreu equilibria with transfers (and with countably additive prices). Transfers are equal to the value of price bubbles on agents' initial portfolio holdings.

Our results imply that there are always sequential equilibria with price bubbles under the wealth constraint if some of the securities are in zero supply or negative prices are permitted. This contradict the often asserted claim that price bubbles cannot arise when agents are infinitely lived (see, for example, Blanchard and Fischer [5] who attribute the claim to Tirole [19]). It is also in contrast to recent results of Montruccio and Privileggi [16] who show that price bubbles are a

"marginal phenomenon" in representative agent economies. Of course, examples of equilibrium price bubbles have been known before, but our results demonstrate their general existence with zero supply or negative prices. Santos and Woodford [18] proved that price bubbles cannot exist when all securities are in strictly positive supply and prices are positive.

We consider an alternative portfolio feasibility constraint which requires that the value of borrowing at normalized security prices be bounded from below. We call portfolio strategies satisfying this constraint *essentially bounded* portfolio strategies. This feasibility constraint has a remarkable property that there cannot be price bubbles in sequential equilibrium regardless of the supply of the securities. We prove that exact implementation of Arrow-Debreu equilibria with countably additive prices (without transfers) holds with essentially bounded portfolio strategies.

It should be emphasized that Arrow-Debreu equilibria that can be implemented by sequential trading of infinitely-lived securities must have countably additive prices. It has been known since Bewley [4] that for some class of economies Arrow-Debreu equilibrium prices may not be countably additive (for an example, see Huang and Werner [10]). Our results indicate that those equilibria cannot be implemented by sequential trading (except when the equilibrium allocation can also be supported by countable additive prices).

The concept of Arrow-Debreu equilibrium underlying our analysis is due to Peleg and Yaari [17]. Equilibrium prices assign finite values to consumption plans that are positive and do not exceed the aggregate endowment, but may or may not assign finite values to other consumption plans. The Peleg and Yaari approach should be contrasted with a more standard approach, first proposed by Debreu [6] (see also Bewley [4]), where the consumption space and the price space are a dual pair of topological vector spaces. Under this second approach, equilibrium prices assign finite values to all consumption plans in the consumption space. Sufficient conditions for the existence of Peleg-Yaari equilibria with countably additive prices can be found in Peleg and Yaari [17]. Aliprantis, Brown and Burkinshaw [1, 2]

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provide an analysis of Peleg-Yaari equilibria in general consumption spaces.

The paper is organized as follows: In section 2 we provide specification of time and uncertainty. In section 3 we introduce the notion of Arrow-Debreu equilibrium and in section 4 we define a sequential equilibrium in security markets. We assume that there is a finite number of infinitely-lived agents and a finite number of infinitely-lived securities available for trade at every date. In sections 5 and 6 we state and prove our basic implementation results. In section 7 we present an example that illustrates our results. We conclude with a discussion of alternative portfolio constraints in section 8.

2. Time and Uncertainty

Time is discrete with infinite horizon and indexed by $t = 0, 1 \ldots$. Uncertainty is described by a set $\mathcal{S}$ of states of the world and an increasing sequence of finite partitions $\{\mathcal{F}_t\}_{t=0}^{\infty}$ of $\mathcal{S}$. A state $s \in \mathcal{S}$ specifies a complete history of the environment from date 0 to the infinite future. The partition $\mathcal{F}_t$ specifies sets of states that can be verified by the information available at date t. An element $s^t \in \mathcal{F}_t$ is called a date-t event. We take $\mathcal{F}_0 = \mathcal{S}$ so that there is no uncertainty at date 0.

This description of the uncertain environment can be interpreted as an event tree. An event $s^t \in \mathcal{F}_t$ at date t identifies a node of the event tree. The unique date 0 event s^0 is the root node of the event tree. The set of all events at all dates is denoted by $\mathcal{E}$. For each node s^t there is a set of immediate successors and (with exception of the root node) a unique predecessor. The unique predecessor of s^t is a date-$(t-1)$ event $s_-^t \in \mathcal{F}_{t-1}$ such that $s^t \subset s_-^t$. An immediate successor of s^t is a date-$(t+1)$ event s^{t+1} such that $s^{t+1} \in \mathcal{F}_{t+1}$ and $s^{t+1} \subset s^t$. The set of all immediate successors of s^t is denoted by $\mathcal{F}_{t+1}(s^t)$ and the number of immediate successors s^t by $\kappa(s^t)$. We assume that $\sup_{s^t \in \mathcal{E}} \kappa(s^t) < \infty$, and denote that supremum by K.

The set of all date-τ successor events of s^t for $\tau > t$, that is all date-τ events $s^\tau \in \mathcal{F}_\tau$ with $s^\tau \subset s^t$, is denoted by $\mathcal{F}_\tau(s^t)$. The set of successor events of s^t at all dates after t is denoted by $\mathcal{E}^+(s^t)$. We also write $\mathcal{E}(s^t) \equiv \{s^t\} \cup \mathcal{E}^+(s^t)$.

3. Arrow-Debreu Equilibrium

There is a single consumption good. A consumption plan is a scalar-valued process adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Consumption plans are restricted to lie in a linear space C of adapted processes. Our primary choice of the consumption space C is the space of all adapted processes (which can be identified with $\mathcal{R}^{\infty}$). The cone of nonnegative processes in C is denoted by C_+; a typical element of C is denoted by $c = \{c(s^t)\}_{s^t \in \mathcal{E}}$.

There are I consumers. Each consumer i has the consumption set C_+, a strictly increasing and complete preference $\preceq^i$ on C_+, and an initial endowment $\omega^i \in C_+$. The aggregate endowment $\bar{\omega} \equiv \sum_i \omega^i$ is assumed positive, that is, $\bar{\omega} \geq 0$.

The standard notion of an Arrow-Debreu general equilibrium is extended to our setting with infinitely many dates as follows: Prices are described by linear functional P which is positive and well-defined (i.e., finite valued) on each consumer's initial endowment. We call such functional a *pricing functional*. It follows that a pricing functional is well-defined on the aggregate endowment $\bar{\omega}$ and, since it is positive, also on each attainable consumption plan, that is, on each c satisfying $0 \leq c \leq \bar{\omega}$. It may or may not be well-defined on the entire space C.

The price of one unit consumption in event s^t under pricing functional P is $p(s^t) \equiv P(e(s^t))$, where $e(s^t)$ denotes the consumption plan equal to 1 in event s^t at date t and zero in all other events and all other dates. A pricing functional P is *countably additive* if and only if $P(c) = \sum_{\mathcal{E}} p(s^t)c(s^t)$ for every c for which $P(c)$ is well-defined.

An *Arrow-Debreu equilibrium* is a pricing functional P and a consumption allocation $\{c^i\}_{i=1}^{I}$ such that c^i maximizes consumer i's preference $\preceq^i$ subject to $P(c) \leq P(\omega^i)$ and $c \in C_+$, and markets clear, that is $\sum_i c^i = \sum_i \omega^i$. An equilibrium pricing functional is normalized so that $p(s^0) = 1$.

As noted in the introduction, this concept of Arrow-Debreu equilibrium is due to Peleg and Yaari [17] who also provide sufficient conditions for the existence of an equilibrium with countably additive pricing functional when the consumption

space is $C = \mathcal{R}^\infty$. The conditions are the standard monotonicity and convexity of preferences, as well as continuity of preferences in the product topology.

We will also need the notion of an equilibrium with transfers. For given transfers $\{\epsilon^i\}_{i=1}^I$, where $\epsilon^i \in \mathcal{R}$ and $\sum_i \epsilon^i = 0$, a pricing functional P (with $p(s^0) = 1$) and a consumption allocation $\{c^i\}_{i=1}^I$ are an *Arrow-Debreu equilibrium with transfers* if c^i maximizes consumer i's preference $\preceq^i$ subject to $P(c) \leq P(\omega^i) + \epsilon^i$ and $c \in C_+$, and markets clear. Peleg and Yaari [17] conditions also imply that an Arrow-Debreu equilibrium with transfers exists for small transfers.

4. Sequential Equilibrium and Price Bubbles

We consider J infinitely-lived securities traded at every date. We assume that the number of securities is greater than or equal to the number of immediate successors of every event, that is, $J \geq K$. Each security j is specified by a dividend process d_j which is adapted to $\{\mathcal{F}_t\}_{t=0}^\infty$ and nonnegative. The ex-dividend price of security j in event s^t is denoted by $q_j(s^t)$, and q_j is the price process of security j. Portfolio strategy θ specifies a portfolio of J securities $\theta(s^t)$ held after trade in each event s^t. The payoff of portfolio strategy θ in event s^t for $t \geq 1$ at a price process q is

$$z(q, \theta)(s^t) \equiv [q(s^t) + d(s^t)]\theta(s^t_-) - q(s^t)\theta(s^t). \tag{1}$$

Each consumer i has an initial portfolio $\alpha^i \in \mathcal{R}^J$ at date 0. The dividend stream $\alpha^i d$ on initial portfolio constitutes one part of consumer i's endowment. The rest is $y^i \in C$ and becomes available to the consumer at each date in every event. Thus, it holds

$$\omega^i(s^t) = y^i(s^t) + \alpha^i d(s^t) \quad \forall s^t. \tag{2}$$

The supply of securities is $\bar{\alpha} = \sum_i \alpha^i$, and the adjusted aggregate endowment of goods is $\bar{y} = \sum_i y^i$. We assume that $\bar{\alpha} \geq 0$.

Consumers face feasibility constraints when choosing their portfolio strategies. Such constraints are necessary to prevent consumers from using Ponzi schemes. In the definition of sequential equilibrium the set of feasible portfolio strategies of

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consumer i is Θ^i. Specific feasibility constraints will be introduced in sections 5 and 6.

A *sequential equilibrium* is a price process q and consumption-portfolio allocation $\{c^i, \theta^i\}_{i=1}^I$ such that:

(i) for each i, consumption plan c^i and portfolio strategy θ^i maximize $\preceq^i$ subject to

$$c(s^0) + q(s^0)\theta(s^0) \leq y^i(s^0) + q(s^0)\alpha^i,$$
$$c(s^t) \leq y^i(s^t) + z(q, \theta)(s^t) \quad \forall s^t \neq s^0,$$
$$c \in C_+, \ \theta \in \Theta^i;$$

(ii) markets clear, that is

$$\sum_i c^i(s^t) = \bar{y}(s^t) + \bar{\alpha}d(s^t), \quad \sum_i \theta^i(s^t) = \bar{\alpha}, \quad \forall s^t$$

Security price process q is *one-period arbitrage free* in event s^t if there does not exist a portfolio $\theta(s^t)$ such that $[q(s^{t+1}) + d(s^{t+1})]\theta(s^t) \geq 0$ for every $s^{t+1} \in \mathcal{F}_{t+1}(s^t)$ and $q(s^t)\theta(s^t) \leq 0$, with at least one strict inequality.[1] It is well known that if q is arbitrage free in every event, then there exist a sequence of strictly positive numbers $\{\pi_q(s^t)\}_{s^t \in \mathcal{E}}$ with $\pi_q(s^0) = 1$ such that

$$\pi_q(s^t)q_j(s^t) = \sum_{s^{t+1} \in \mathcal{F}_{t+1}(s^t)} \pi_q(s^{t+1})\left[q_j(s^{t+1}) + d_j(s^{t+1})\right] \quad \forall s^t, j. \tag{3}$$

We call such π_q a system of *event prices* associated with q.

Security markets are *one-period complete* in event s^t at prices q if the one-period payoff matrix $[q(s^{t+1}) + d(s^{t+1})]_{s^{t+1} \in \mathcal{F}_{t+1}(s^t)}$ has rank equal to $\kappa(s^t)$. Security markets are *complete* at q if they are one-period complete at every event. Of course, the assumed condition that $J \geq K$ is necessary for markets to be complete.

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[1]Note that one-period arbitrage is defined without any reference to the portfolio feasibility constraint. Of the two constraints considered in this paper, the constraint of essentially bounded strategies (Section 6) does not restrict portfolio holdings in any single event while the wealth constraint (Section 5) does. Yet, it remains true that there cannot be a one-period arbitrage in sequential equilibrium under the wealth constraint (see Santos and Woodford [18]).

If markets are complete at q, then for each event s^t there exists a portfolio strategy that has payoff equal to one at s^t, zero in every other event and involves no portfolio holding after date t. If q is one-period arbitrage free, then the date-0 price of that portfolio strategy is $\pi_q(s^t)$ which justifies the term event price.

Suppose that security prices q are one-period arbitrage free and that markets are complete at q. Then the *present value* of security j at s^t can be defined using event prices as

$$\frac{1}{\pi_q(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau) d_j(s^\tau). \tag{4}$$

If the price of security j is nonnegative in every event, then the sum (4) is finite for every s^t. To see this, we use (3) recursively to obtain

$$q_j(s^t) = \sum_{\tau=t+1}^{T} \sum_{s^\tau \in \mathcal{F}_\tau(s^t)} \frac{\pi_q(s^\tau)}{\pi_q(s^t)} d_j(s^\tau) + \sum_{s^T \in \mathcal{F}_T(s^t)} \frac{\pi_q(s^T)}{\pi_q(s^t)} q_j(s^T) \tag{5}$$

for each s^t, and for any $T > t$. If $q_j(s^t) \geq 0$, then (5) implies that

$$q_j(s^t) \geq \frac{1}{\pi_q(s^t)} \sum_{\tau=t+1}^{T} \sum_{s^\tau \in \mathcal{F}_\tau(s^t)} \pi_q(s^\tau) d_j(s^\tau) \tag{6}$$

for every s^t and $T > t$. Taking the limit on the right hand side of (6) as T goes to infinity and using $d_j(s^\tau) \geq 0$, we obtain that the present value (4) is less than or equal to the price of the security.

We do not exclude the possibility of security prices being negative. Absence of one-period arbitrage does not imply that security prices are nonnegative even if dividends are nonnegative.[2] A way to exclude negative security prices is to assume free disposal of securities (see Santos and Woodford [18]).[3]

If the present value of a security is finite, then the difference between the price and the present value is the *price bubble* on that security. We denote the price

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[2] We show later in the paper that negative security prices are possible in sequential equilibrium under the wealth constraint but not with essentially bounded portfolio strategies.

[3] The assumption of free disposal is prohibitively restrictive for many securities. For example, futures markets would not exist if futures contracts could be freely disposed.

bubble (associated with q) on security j in event s^t by $\sigma_{qj}(s^t)$. That is

$$\sigma_{qj}(s^t) \equiv q_j(s^t) - \frac{1}{\pi_q(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau)d_j(s^\tau). \tag{7}$$

Note that if the price of security j is nonnegative in every event, then $0 \le \sigma_{qj}(s^t) \le q_j(s^t)$ for every s^t. Also, if the present value of security j is finite and $\sigma_{qj}(s^t) \ge 0$ for every s^t, then $q_j(s^t) \ge 0$ for every s^t.

For use later, we note that (5) and (7) imply that

$$\sigma_{qj}(s^t) = \frac{1}{\pi_q(s^t)} \sum_{s^{t+1} \in \mathcal{F}_{t+1}(s^t)} \pi_q(s^{t+1})\sigma_{qj}(s^{t+1}), \tag{8}$$

and also that

$$\sigma_{qj}(s^t) = \lim_{T \to \infty} \frac{1}{\pi_q(s^t)} \sum_{s^T \in \mathcal{F}_T} \pi_q(s^T)q_j(s^T). \tag{9}$$

for each s^t.

Whether nonzero price bubbles can exist in a sequential equilibrium depends crucially on the form of portfolio feasibility constraints (see Huang and Werner [10]). Under the wealth constraint (Section 5) nonzero equilibrium price bubbles are possible but they are not possible under the constraint of essentially bounded portfolio strategies (Section 6).

5. Implementation with the Wealth Constraint.

A frequently used portfolio feasibility constraint is the wealth constraint. It applies to complete security markets where event prices can be uniquely defined.[4] It prohibits a consumer from borrowing more than the present value of his future endowment. Formally, portfolio strategy θ satisfies the *wealth constraint* if

$$q(s^t)\theta(s^t) \ge -\frac{1}{\pi_q(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau)y^i(s^\tau) \qquad \forall s^t, \tag{10}$$

where π_q is the event price system (assumed unique) associated with q. We refer to a sequential equilibrium in which every consumer's set of feasible portfolio strategies is defined by (10) as a sequential equilibrium under the wealth constraint.

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[4]Santos and Woodford [18] extend the wealth constraint to incomplete markets.

We begin with two theorems that establish equivalence between countably additive Arrow-Debreu equilibria and sequential equilibria with no price bubbles. All proofs have been relegated to the Appendix.

Theorem 5.1. *Let consumption allocation $\{c^i\}_{i=1}^I$ and pricing functional P be an Arrow-Debreu equilibrium. If P is countably additive, $P(d_j) < \infty$ for each j, and security markets are complete at prices q given by*

$$q_j(s^t) = \frac{1}{p(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau)d_j(s^\tau), \qquad \forall s^t, j, \tag{11}$$

then there exists a portfolio allocation $\{\theta^i\}_{i=1}^I$ such that q and the allocation $\{c^i, \theta^i\}_{i=1}^I$ are a sequential equilibrium under the wealth constraint.

Theorem 5.1 says that an Arrow-Debreu equilibrium with countably additive pricing can be implemented by sequential trading under the wealth constraint provided that security markets are complete at prices defined by the present value of future dividends (and thus with no price bubbles). The assumed condition that there are more securities than immediate successors of every event $(J \geq K)$ is necessary for market completeness.

The implementation of countably additive Arrow-Debreu equilibria by sequential trading with no price bubbles is exact.

Theorem 5.2 *Let security prices q and consumption-portfolio allocation $\{c^i, \theta^i\}_{i=1}^I$ be a sequential equilibrium under the wealth constraint. If security markets are complete at q and price bubbles are zero, i.e., $\sigma_q = 0$, then consumption allocation $\{c^i\}_{i=1}^I$ and the pricing functional P given by*

$$P(c) = \sum_{s^t \in \mathcal{E}} \pi_q(s^t)c(s^t) \tag{12}$$

are an Arrow-Debreu equilibrium.

Our next results show that, in general, there exist sequential equilibria under the wealth constraint with nonzero price bubbles, and that they correspond to countably additive Arrow-Debreu equilibria with transfers.

Theorem 5.3. *Let consumption allocation $\{c^i\}_{i=1}^I$ and pricing functional P be an Arrow-Debreu equilibrium with transfers $\{\epsilon^i\}_{i=1}^I$ such that $\epsilon^i = \rho(s^0)\alpha^i$ for some $\rho(s^0) \in \mathcal{R}^J$ with $\rho(s^0)\bar{\alpha} = 0$. If P is countably additive, $P(d_j) < \infty$ for each j, and security markets are complete at prices q given by*

$$q_j(s^t) = \frac{1}{p(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau)d_j(s^\tau) + \rho_j(s^t), \qquad \forall s^t, j, \tag{13}$$

where $\rho(s^t) \in \mathcal{R}^J$ satisfies $\rho(s^t)\bar{\alpha} = 0$ for every s^t and

$$\rho_j(s^t) = \frac{1}{p(s^t)} \sum_{s^{t+1} \in \mathcal{F}_{t+1}(s^t)} p(s^{t+1})\rho_j(s^{t+1}), \qquad \forall s^t, j, \tag{14}$$

then there exists a portfolio allocation $\{\theta^i\}_{i=1}^I$ such that q and allocation $\{c^i, \theta^i\}_{i=1}^I$ are a sequential equilibrium under the wealth constraint.

Theorem 5.3 says that every Arrow-Debreu equilibrium allocation with transfers that are proportional to initial portfolios and with countably additive pricing can be implemented by sequential trading under the wealth constraint, provided that security markets are complete. Proportionality of transfers to initial portfolios, i.e., $\epsilon^i = \rho(s^0)\alpha^i$ for some $\rho(s^0)$, implies that a consumer whose initial portfolio is zero, must have zero transfer. If all consumers have zero initial portfolios, then only Arrow-Debreu equilibrium without transfers can be implemented in sequential markets.

Security prices in Theorem 5.3 have price bubbles equal to ρ. There is considerable freedom in choosing price bubbles in the construction of prices (13). The only constraints on ρ are the "martingale property" (14) and the condition of zero bubble on the supply of securities. This freedom in choosing price bubbles can be used to assure that security markets are complete at prices (13) (see the example of Section 7).

If one is willing to restrict attention to positive security prices and therefore to positive price bubbles, the scope of Theorem 5.3 is restricted as ρ has to be positive. With ρ positive condition $\rho(s^0)\bar{\alpha} = 0$ implies that $\rho_j(s^0) = 0$ for every

security j with $\bar{\alpha}_j > 0$. Then (14) implies that $\rho_j(s^t) = 0$ for such j, for every s^t. Thus, there cannot be positive price bubbles in sequential equilibrium on a security with strictly positive supply. This has been demonstrated by Santos and Woodford [18]. If all securities are in strictly positive supply, then only Arrow-Debreu equilibrium without transfers can be implemented in sequential markets with positive prices.

The implementation of countably additive Arrow-Debreu equilibria with transfers is exact.

Theorem 5.4. *Let security prices q and consumption-portfolio allocation $\{c^i, \theta^i\}_{i=1}^I$ be a sequential equilibrium under the wealth constraint. If security markets are complete at q and $\sum_{s^t \in \mathcal{E}} \pi_q(s^t) d_j(s^t) < \infty$ for each j, then consumption allocation $\{c^i\}_{i=1}^I$ and the pricing functional P given by*

$$P(c) = \sum_{s^t \in \mathcal{E}} \pi_q(s^t) c(s^t) \tag{15}$$

are an Arrow-Debreu equilibrium with transfers $\{\sigma_q(s^0)\alpha^i\}_{i=1}^I$. It holds $\sigma_q(s^0)\bar{\alpha} = 0$.

We emphasize that Arrow-Debreu equilibria (with or without transfers) that can be implemented in sequential markets under the wealth constraint have countably additive pricing. There are economies in which Arrow-Debreu equilibrium pricing functional is not countably additive (see Huang and Werner [10]). These equilibria cannot be implemented in sequential markets under the wealth constraint.

6. Implementation with Essentially Bounded Portfolios.

We propose in this section a portfolio feasibility constraint under which only Arrow-Debreu equilibria without transfers can be implemented in sequential markets and price bubbles cannot arise.

Portfolio strategy θ is *bounded from below* if

$$\inf_{s^t, j} \theta_j(s^t) > -\infty \tag{16}$$

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Portfolio strategy θ is *essentially bounded from below* at q if there exists a bounded from below portfolio strategy b such that

$$q(s^t)\theta(s^t) \geq q(s^t)b(s^t) \qquad \forall s^t. \tag{17}$$

We refer to a (essentially) bounded from below portfolio strategies simply as (essentially) bounded portfolio strategy. Of course, every bounded portfolio strategy is essentially bounded but the converse is not true (unless there is a single security). The set of essentially bounded portfolio strategies is a convex cone.

If security price vector $q(s^t)$ is positive and nonzero for every event s^t, then portfolio strategy θ is essentially bounded if and only if

$$\inf_{s^t} \bar{q}(s^t)\theta(s^t) > -\infty, \tag{18}$$

where $\bar{q}(s^t) \equiv q(s^t)/\sum_j q_j(s^t)$ is the normalized security price vector.[5]

We refer to sequential equilibrium in which each consumer's set of feasible portfolios is the set of essentially bounded portfolio strategies (17) as a sequential equilibrium with essentially bounded portfolios.

It is crucial for the results in this section that the portfolio feasibility constraint is stated in the form (17). Neither the bounded borrowing constraint $\inf_{s^t} q(s^t)\theta(s^t) > -\infty$, nor (16) deliver the same results. Only if there is a single security, (17) and (16) are equivalent, and in that sense the results of this section extend Theorem 9.1 in Huang and Werner [10].

Before presenting the implementation results we prove that there cannot be a nonzero price bubble in sequential equilibrium with essentially bounded portfolios. All proofs in this section have been relegated to the Appendix.

Theorem 6.1. *If q is a sequential equilibrium price process with essentially bounded portfolios and if security markets are complete at q, then $q(s^t) \geq 0$ and $\sigma_q(s^t) = 0$ for every s^t.*

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[5] If θ satisfies (17) and $q \geq 0$, then $q(s^t)\theta(s^t) \geq [\sum_j q_j(s^t)]\underline{b}$, where $\underline{b} = \inf_{s^t,j} b_j(s^t)$. Hence $\bar{q}(s^t)\theta(s^t)$ is bounded below. Conversely, if θ satisfies (18), then $\bar{q}(s^t)\theta(s^t) \geq B$ for some $B \in \mathcal{R}$. With $b_j(s^t) = B$ for each j and s^t, θ satisfies (17).

That the price of each security has to be positive follows from the fact that a portfolio strategy of short-selling the security and never buying it back is bounded. If the price were negative, then each consumer could short-sell the security (and do so at an arbitrary scale) and make an arbitrage profit. This is incompatible with an equilibrium. A similar arbitrage argument implies that price bubbles are zero. For each security there is an essentially bounded portfolio strategy with initial investment equal to the negative of the price bubble and zero payoff in every future event. If the price bubble were positive (it cannot be negative as shown in Section 4) each consumer could make an arbitrage profit of arbitrary scale. A detailed proof can be found in the Appendix.

The following two theorems demonstrate that countably additive Arrow-Debreu equilibria (without transfers) can be implemented by sequential trading with essentially bounded portfolios in exact fashion.

Theorem 6.2. *Let consumption allocation $\{c^i\}_{i=1}^I$ and pricing functional P be an Arrow-Debreu equilibrium. If P is countably additive, $P(d_j) < \infty$ for each j, security markets are complete at prices q given by*

$$q_j(s^t) = \frac{1}{p(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau) d_j(s^\tau), \qquad \forall s^t, j, \tag{19}$$

and there exists an essentially bounded portfolio strategy η such that

$$-\frac{1}{p(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau) \bar{y}(s^t) \geq q(s^t) \eta(s^t), \qquad \forall s^t, \tag{20}$$

then there exists a portfolio allocation $\{\theta^i\}_{i=1}^I$ such that q and the allocation $\{c^i, \theta^i\}_{i=1}^I$ are a sequential equilibrium with essentially bounded portfolios.

Condition (20) says that it is feasible to borrow an amount greater than or equal to the present value of aggregate future endowment using an essentially bounded portfolio strategy. Equivalently, it is feasible for each consumer to borrow the present value of his endowment using an essentially bounded portfolio strategy. Under condition (20), the set of all essentially bounded portfolio strategies includes all strategies satisfying the wealth constraint.

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Theorem 6.3. *Let security prices q and consumption-portfolio allocation $\{c^i, \theta^i\}_{i=1}^I$ be a sequential equilibrium with essentially bounded portfolios. If security markets are complete at q and there exists an essentially bounded portfolio strategy η such that*

$$-\frac{1}{\pi_q(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau) \bar{y}(s^t) \geq q(s^t) \eta(s^t), \qquad \forall s^t, \tag{21}$$

then consumption allocation $\{c^i\}_{i=1}^I$ and pricing functional P given by

$$P(c) = \sum_{s^t \in \mathcal{E}} \pi_q(s^t) c(s^t) \tag{22}$$

are an Arrow-Debreu equilibrium.

One can show that for condition (21) or (20) to hold it is sufficient that there exists a bounded from above and from below portfolio strategy b such that $\bar{y}(s^t) \leq z(q, b)(s^t)$ for all s^t. For this latter condition, it is sufficient that $\bar{y}$ is bounded relative to d, that is, that $\bar{y}(s^t) \leq \gamma d(s^t)$ for some $\gamma \in \mathcal{R}^J$, for all s^t.

7. Example.

We consider an infinite-time binomial event-tree. Event s^t at date t has two immediate successors (s^t, up) and $(s^t, down)$ at date $t+1$. Probabilities of events are specified by conditional probabilities $\mu(up|s^t) = \mu(down|s^t) = \frac{1}{2}$.

There are two consumers with expected utility functions

$$u^i(c) = \sum_{t=0}^\infty \beta^t E[ln(c_t)], \tag{23}$$

for $i = 1, 2$, where expectation is taken with respect to μ, and $0 < \beta < 1$. Their endowments are

$$\omega^1(s^t, up) = A, \quad \omega^1(s^t, down) = B, \tag{24}$$

and

$$\omega^2(s^t, up) = B, \quad \omega^2(s^t, down) = A, \tag{25}$$

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for each s^t with $t \geq 0$, and

$$\omega^1(s^0) = \omega^2(s^0) = \frac{A + B}{2}. \tag{26}$$

The aggregate endowment is $A + B$ – risk-free, and constant over time.

The unique Arrow-Debreu equilibrium consists of allocation $\{c^1, c^2\}$ given by

$$c^1(s^t) = c^2(s^t) = \frac{A + B}{2}, \quad \forall s^t, \tag{27}$$

supported by a countably additive pricing functional P given by

$$p(s^t) = \beta^t \mu(s^t). \tag{28}$$

Next, we consider security markets with two infinitely-lived securities. Security 1 has risk-free dividends given by

$$d_1(s^t) = 1 \quad \forall s^t, \quad t \geq 1. \tag{29}$$

Security 2 has state-dependent dividends given by

$$d_2(s^t, up) = u, \qquad d_2(s^t, down) = d, \tag{30}$$

for each s^t with $t \geq 0$. We assume that $u > d$. Initial portfolios are $\alpha^1 = (1, 1)$ and $\alpha^2 = (-1, 0)$. Adjusted goods endowments are $y^1 = \omega^1 - d_1 - d_2$ and $y^2 = \omega^2 + d_1$.

Theorem 5.1 says that the Arrow-Debreu equilibrium consumption allocation (27) can be implemented as a sequential equilibrium under the wealth constraint. Equilibrium security prices obtain from (11) and are

$$q_1(s^t) = \sum_{\tau=1}^{\infty} \beta^\tau = \frac{\beta}{1 - \beta} \tag{31}$$

and

$$q_2(s^t) = \frac{u + d}{2} \sum_{\tau=1}^{\infty} \beta^\tau = \frac{u + d}{2} \frac{\beta}{1 - \beta}, \tag{32}$$

for every s^t. At these prices security markets are complete. Further, there are no price bubbles.

For small enough transfers ϵ^1, ϵ^2 (with $\epsilon^1 + \epsilon^2 = 0$), there is an Arrow-Debreu equilibrium with transfers. It consists of consumption allocation

$$\tilde{c}^1(s^t) = \frac{A+B}{2} + \epsilon^1(1-\beta), \quad \forall s^t, \tag{33}$$

and

$$\tilde{c}^2(s^t) = \frac{A+B}{2} + \epsilon^2(1-\beta), \quad \forall s^t, \tag{34}$$

supported by the countably additive pricing functional P given by (28).

Theorem 5.3 implies that consumption allocation (33 - 34) of the Arrow-Debreu equilibrium with transfers can be implemented as a sequential equilibrium under the wealth constraint, provided that transfers are proportional to initial portfolios. Transfers ϵ^1 and ϵ^2 have to satisfy $\epsilon^1 = \rho(s^0)\alpha^1$ and $\epsilon^2 = \rho(s^0)\alpha^2$ for some $\rho(s^0) = (\rho_1(s^0), \rho_2(s^0)) \in \mathcal{R}^2$ with $\rho(s^0)\bar{\alpha} = 0$. Since $\bar{\alpha} = (0,1)$, this implies $\rho_2(s^0) = 0$, $\epsilon^1 = \rho_1(s^0)$ and $\epsilon^2 = -\rho_1(s^0)$, with no restriction on $\rho_1(s^0)$, and therefore no restriction (beyond $\epsilon^1 + \epsilon^2 = 0$) on transfers. Thus every Arrow-Debreu equilibrium with transfers can be implemented.

Equilibrium security prices can be obtained from (13). They are

$$\tilde{q}_1(s^t) = \sum_{\tau=1}^{\infty} \beta^\tau + \rho_1(s^t) = \frac{\beta}{1-\beta} + \rho_1(s^t), \tag{35}$$

and

$$\tilde{q}_2(s^t) = \frac{u+d}{2} \sum_{\tau=1}^{\infty} \beta^\tau + \rho_2(s^t) = \frac{u+d}{2} \frac{\beta}{1-\beta} + \rho_2(s^t), \tag{36}$$

where $\rho(s^t) = (\rho_1(s^t), \rho_2(s^t))$ is chosen for each s^t so as to satisfy

$$\rho(s^0) = (\epsilon^1, 0) \tag{37}$$

$$\rho(s^t) = \beta[\frac{1}{2}\rho(s^t, up) + \frac{1}{2}\rho(s^t, down)]. \tag{38}$$

Further, the sequence of price bubbles ρ has to be selected so that security markets are complete at prices $\tilde{q}$. Markets are complete whenever the one-period 2-by-2 payoff matrix $[\tilde{q}(s^{t+1}) + d(s^{t+1})]_{s^{t+1} \in \mathcal{F}_{t+1}(s^t)}$ has full rank for every s^t. If we restrict price bubbles to be positive, then necessarily $\rho_2(s^t) = 0$ for every s^t so that there

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can be positive price bubbles only on zero-supply security 1. We recall that negative price bubbles lead eventually to negative security prices in some events.

The results of Section 6 imply that only the Arrow-Debreu equilibrium allocation (27) can be implemented as sequential equilibrium with essentially bounded portfolios. Equilibrium security prices are those of (31 - 32). Arrow-Debreu equilibrium allocations with transfers (33 - 34) cannot be implemented. In particular, security prices (35 - 36) are not equilibrium prices with essentially bounded portfolios since they permit arbitrage opportunities (see Theorem 6.1 and the discussion thereafter).

8. Other Portfolio Constraints.

An often used feasibility constraint on portfolio strategies is the *transversality condition*. In our setting the transversality condition is written as

$$\liminf_{T \to \infty} \sum_{s^T \in \mathcal{F}_T(s^t)} \pi_q(s^T) q(s^T) \theta(s^T) \geq 0 \quad \forall s^t. \tag{39}$$

Hernandez and Santos [9] proved that consumers' budget sets in sequential markets are the same under the wealth constraint and the transversality condition, as long as $\sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t) < \infty$. Therefore, all implementation results of Section 5 remain valid when the wealth constraint is replaced by the transversality condition (and under an additional assumption that $\sum_{s^t \in \mathcal{E}} \pi_q(s^t) \bar{y}(s^t) < \infty$).

Hernandez and Santos [9] and Magill and Quinzii [15] provide other specifications of portfolio constraints that lead to the same budget sets as the wealth constraint. For further discussion of equivalent portfolio constraints in the setting with one-period-lived securities, see Florenzano and Gourdel [8], Magill and Quinzii [15], and Levine and Zame [13].

Appendix.

We start by proving a lemma concerning relation between budget sets in sequential markets under the wealth constraint and in Arrow-Debreu markets. We recall that π_q and σ_q denote the systems of event prices and price bubbles (respectively) associated with given security prices q. Further, p denotes the system of event prices associated with given pricing functional P.

Let $B_w(q; y^i, \alpha^i)$ denote the set of budget feasible consumption plans in sequential markets at prices q under the wealth constraint, when goods endowment is y^i and initial portfolio is α^i. That is, $c \in B_w(q; y^i, \alpha^i)$ if $c \in C_+$ and there exists a portfolio strategy θ such that

$$
\begin{aligned}
c(s^0) + q(s^0)\theta(s^0) &\leq y^i(s^0) + q(s^0)\alpha^i, \\
c(s^t) &\leq y^i(s^t) + z(q, \theta)(s^t) \qquad \forall s^t \neq s^0, \\
q(s^t)\theta(s^t) &\geq -\frac{1}{\pi_q(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau) y^i(s^\tau) \qquad \forall s^t.
\end{aligned}
\tag{40}
$$

Let $B_{AD}(P; \omega^i, \epsilon^i)$ denote the set of budget feasible consumption plans in Arrow-Debreu markets at P when endowment is ω^i and transfer is ϵ^i. That is, $c \in B_{AD}(P; \omega^i, \epsilon^i)$ if $c \in C_+$ and

$$
P(c) \leq P(\omega^i) + \epsilon^i.
\tag{41}
$$

The budget set with zero transfer $B_{AD}(P; \omega^i, 0)$ is denoted by $B_{AD}(P; \omega^i)$.

Throughout the Appendix, endowments ω^i, y^i and α^i are related by (2), that is,

$$
\omega^i(s^t) = y^i(s^t) + \alpha^i d(s^t) \quad \forall s^t.
\tag{42}
$$

Lemma A.1. *Let P be a countably additive pricing functional and q a market-completing system of security prices. If*

$$
\pi_q(s^t) = p(s^t) \qquad \forall s^t,
\tag{43}
$$

and $P(y^i) < \infty, P(d_j) < \infty$ for each i and j, then

$$B_w(q; y^i, \alpha^i) = B_{AD}(P; \omega^i, \alpha^i \sigma_q(s^0)) \tag{44}$$

Proof: Suppose that $c \in B_w(q; y^i, \alpha^i)$. Multiplying both sides of the budget constraint (40) at s^t by $\pi_q(s^t)$ and summing over all s^t for t ranging from 0 to arbitrary τ, and using (3), we obtain

$$\sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} \pi_q(s^t) c(s^t) + \sum_{s^\tau \in \mathcal{F}_\tau} \pi_q(s^\tau) q(s^\tau) \theta(s^\tau) \leq \sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} \pi_q(s^t) y^i(s^t) + q(s^0) \alpha^i. \tag{45}$$

Adding $\sum_{t=\tau+1}^{\infty} \sum_{s^t \in \mathcal{F}_t} \pi_q(s^t) y^i(s^t)$ to both sides of (45), there results

$$\sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} \pi_q(s^t) c(s^t) + \sum_{s^\tau \in \mathcal{F}_\tau} \left[\pi_q(s^\tau) q(s^\tau) \theta(s^\tau) + \sum_{s^t \in \mathcal{E}^+(s^\tau)} \pi_q(s^t) y^i(s^t) \right]$$
$$\leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t) + q(s^0) \alpha^i. \tag{46}$$

The sum $\sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t)$ is finite by assumption. If the use is made of the wealth constraint, (46) implies that

$$\sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} \pi_q(s^t) c(s^t) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t) + q(s^0) \alpha^i \tag{47}$$

Taking limits in (47) as τ goes to infinity yields

$$\sum_{s^t \in \mathcal{E}} \pi_q(s^t) c(s^t) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t) + q(s^0) \alpha^i. \tag{48}$$

Since $\sum_{s^t \in \mathcal{E}} \pi_q(s^t) d_j(s^t)$ is assumed finite for every j, the price bubble $\sigma_q(s^0)$ is well-defined. If the use is made of (7) and (42), inequality (48) can be written as

$$\sum_{s^t \in \mathcal{E}} \pi_q(s^t) c(s^t) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t) \omega^i(s^t) + \alpha^i \sigma_q(s^0), \tag{49}$$

or simply as

$$P(c) \leq P(\omega^i) + \alpha^i \sigma_q(s^0). \tag{50}$$

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Thus $c \in B_{AD}(P; \omega^i, \alpha^i \sigma_q(s^0))$.

Suppose now that $c \in B_{AD}(P; \omega^i, \alpha^i \sigma_q(s^0))$. Since security markets are complete at q, for each s^t there exists portfolio $\theta(s^t)$ such that

$$[q(s^{t+1}) + d(s^{t+1})]\theta(s^t) = \sum_{s^\tau \in \mathcal{E}(s^{t+1})} \frac{\pi_q(s^\tau)}{\pi_q(s^{t+1})}[c(s^\tau) - y^i(s^\tau)] \qquad \forall s^{t+1} \in \mathcal{F}_{t+1}(s^t). \tag{51}$$

Note that the sum on the right-hand side of (51) is finite since $\sum_{s^\tau \in \mathcal{E}(s^{t+1})} \pi_q(s^\tau)c(s^\tau) \leq P(c)$. Multiplying both sides of (51) by $\pi_q(s^{t+1})$, summing over all $s^{t+1} \in \mathcal{F}_{t+1}(s^t)$, and using (3), we obtain

$$q(s^t)\theta(s^t) = \sum_{s^\tau \in \mathcal{E}^+(s^t)} \frac{\pi_q(s^\tau)}{\pi_q(s^t)}[c(s^\tau) - y^i(s^\tau)] \qquad \forall s^t. \tag{52}$$

It follows from (51) and (52) that

$$c(s^t) + q(s^t)\theta(s^t) = y^i(s^t) + [q(s^t) + d(s^t)]\theta(s^t_-) \qquad \forall s^t \neq s^0. \tag{53}$$

Thus c and θ satisfy the sequential budget constraint (40) at each $s^t \neq s^0$. To show that the budget constraint at s^0 also holds we use the equivalence of (50) and (48). Equation (52) for s^0 and (48) imply the date-0 budget constraint

$$c(s^0) + q(s^0)\theta(s^0) \leq y^i(s^0) + q(s^0)\alpha^i. \tag{54}$$

Since $c \geq 0$, equation (52) implies that θ satisfies the wealth constraint. Thus $c \in B_w(q; y^i, \alpha^i)$ $\square$

Proof of Theorem 5.1: For the system of security prices q defined by (11), the associated event prices π_q satisfy (43) and price bubbles σ_q are zero. Since $P(\omega^i) < \infty$ and $P(d_j) < \infty$, it follows that $P(y^i) < \infty$. Therefore, Lemma A.1 implies that the budget set in sequential markets at q equals the Arrow-Debreu budget set with zero transfer. Hence, consumption plan c^i is optimal for each i in sequential markets. It remains to be shown that portfolio strategies that generate the optimal consumption plans clear security markets.

Let θ^i be portfolio strategy defined by (51) with c^i, that is, satisfying

$$[q(s^{t+1}) + d(s^{t+1})]\theta^i(s^t) = \sum_{s^\tau \in \mathcal{E}(s^{t+1})} \frac{p(s^\tau)}{p(s^{t+1})}[c^i(s^\tau) - y^i(s^\tau)] \qquad \forall s^{t+1} \in \mathcal{F}_{t+1}(s^t)$$

(55)

for each s^t and each i. Such portfolio strategy generates consumption plan c^i and satisfies i's the wealth constraint. Summing (55) over all i and using $\sum_i c^i = \sum_i \omega^i$ and (42), we obtain

$$[q(s^t) + d(s^t)]\sum_i \theta^i(s^t_-) = \sum_{s^\tau \in \mathcal{E}(s^t)} \frac{p(s^\tau)}{p(s^t)}d(s^\tau)\bar{\alpha} \qquad \forall s^t \neq s^0.$$

(56)

It follows from (11) and (56) that

$$[q(s^t) + d(s^t)]\left[\sum_i \theta^i(s^t_-) - \bar{\alpha}\right] = 0 \qquad \forall s^t \neq s^0.$$

(57)

If there are no securities with redundant one-period payoffs, then (57) implies that

$$\sum_i \theta^i(s^t) = \bar{\alpha} \qquad \forall s^t.$$

(58)

Otherwise, if there are redundant securities, then portfolio strategies $\{\theta^i\}$ can be modified without changing their payoffs so that (58) holds. □

Proof of Theorem 5.2: In a sequential equilibrium under the wealth constraint, the present value $\frac{1}{\pi_q(s^t)}\sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau)y^i(s^\tau)$ must be finite, for otherwise there would not exist an optimal portfolio strategy for consumer i. Therefore $P(y^i) < \infty$. Further, since price bubbles are zero, it follows that $P(d_j) < \infty$ (see Section 4). Lamma A.1 can be applied and it implies the conclusion. □

Proof of Theorem 5.3: If security prices q defined by (13) are market completing, then the associated system of event prices π_q is unique and satisfies (43). The associated price bubbles are $\sigma_q = \rho$. As in the proof of Theorem 5.1, one can show that $P(y^i) < \infty$. It follows from Lemma A.1 that each consumption plan c^i is optimal in sequential markets.

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The proof that portfolio strategies that generate the optimal consumption plans also clear security markets is the same as in Theorem 5.1 with one minor modification. With security prices defined by (13), we obtain the following equation instead of (57):

$$[q(s^t) + d(s^t)] \left[\sum_i \theta^i(s^t_-) - \bar{\alpha} \right] = -\rho(s^t)\bar{\alpha} \qquad \forall s^t \neq s^0. \tag{59}$$

However, since $\rho(s^t)\bar{\alpha} = 0$, equation (57) does hold and the rest of the proof of Theorem 5.1 applies.

Proof of Theorem 5.4: The same argument as in the proof of Theorem 5.2 implies that $P(y^i) < \infty$. It follows immediately from Lemma A.1 that pricing functional P and consumption allocation $\{c^i\}_{i=1}^I$ are an Arrow-Debreu equilibrium with transfers $\{\sigma_q(s^0)\alpha^i\}_{i=1}^I$. That these transfers add up to zero (or, equivalently, price bubble on the supply of securities is zero) follows from Walras' Law

$$\sum_i P(c^i) = \sum_i P(\omega^i) + \sigma_q(s^0)\bar{\alpha} \tag{60}$$

and market-clearing $\sum_i c^i = \sum_i \omega^i$. □

Proof of Theorem 6.1: We first show that $q \geq 0$. Suppose, by contradiction, that $q_j(s^t) < 0$ for some security j and event s^t. Let c^i be equilibrium consumption plan and θ^i equilibrium portfolio strategy of consumer i. Consider a portfolio strategy $\hat{\theta}^i$ that results from holding θ^i and purchasing one share of security j in event s^t and holding it forever. Since θ^i is essentially bounded, $\hat{\theta}^i$ is essentially bounded, too. Further, since $q_j(s^t) < 0$ and $d_j \geq 0$, portfolio strategy $\hat{\theta}^i$ generates a consumption plan that is greater than or equal to c^i in every event and strictly greater in event s^t. This contradicts the optimality of c^i.

We can now assume that $q \geq 0$. It follows from the discussion in Section 4 that the present value (4) of each security is finite and the price bubble is well-defined and nonnegative in every event. To prove that $\sigma_q(s^t) = 0$ for every t it suffices to show that $\sigma_q(s^0) = 0$. The rest follows from (8).

Since security markets are complete at q, for each event s^t and each security j there exists a portfolio $\xi^j(s^t)$ such that

$$[q(s^{t+1}) + d(s^{t+1})]\xi^j(s^t) = \sum_{s^\tau \in \mathcal{E}(s^{t+1})} \frac{\pi_q(s^\tau)}{\pi_q(s^{t+1})} d_j(s^\tau) \qquad \forall s^{t+1} \in \mathcal{F}_{t+1}(s^t). \quad (61)$$

Multiplying both sides of (61) by $\pi_q(s^{t+1})$, summing over all $s^{t+1} \in \mathcal{F}_{t+1}(s^t)$ and using (3) we obtain

$$q(s^t)\xi^j(s^t) = \sum_{s^\tau \in \mathcal{E}^+(s^t)} \frac{\pi_q(s^\tau)}{\pi_q(s^\tau)} d_j(s^\tau). \quad (62)$$

Since $d_j \geq 0$, it follows that $q(s^t)\xi^j(s^t) \geq 0$. Therefore ξ^j is essentially bounded. Using (61) and (62), we obtain

$$[q(s^t) + d(s^t)]\xi^j(s^t_-) - q(s^t)\xi^j(s^t) = d_j(s^t) \qquad \forall s^t \neq s^0, \quad (63)$$

Thus, the payoff of ξ^j equals the dividend d_j, that is $z(q, \xi^j) = d_j$. Date-0 price of ξ^j is the present value of d_j (see (62)).

Let η^j denote a portfolio strategy of selling one share of security j at date 0 and never buying it back. We have

$$z(q, \xi^j + \eta^j)(s^t) = 0 \qquad \forall s^t \neq s^0. \quad (64)$$

and

$$q(s^0)[\xi^j(s^0) + \eta^j(s^0)] = -\sigma_{qj}(s^0). \quad (65)$$

Consider portfolio strategy $\hat{\theta}^i = \theta^i + \xi^j + \eta^j$. Since strategies θ^i, ξ^j and η^j are essentially bounded, $\hat{\theta}^i$ is essentially bounded, too. If $\sigma_{qj}(s^0) > 0$, then $\hat{\theta}^i$ generates a consumption plan that is strictly greater than c^i at date 0 and equal to c^i in all future events. This would contradict optimality of c^i. Therefore $\sigma_{qj}(s^0) = 0$. $\square$

Before proving Theorems 6.2 and 6.3 we establish a lemma concerning relation between budget sets in sequential markets with essentially bounded portfolio strategies and in Arrow-Debreu markets.

Let $B_b(q; y^i, \alpha^i)$ denote the set of budget feasible consumption plans in sequential markets at prices q with essentially bounded portfolio strategies, when goods

endowment is y^i and initial portfolio is α^i. That is, $c \in B_b(q; y^i, \alpha^i)$ if $c \in C_+$ and there exists an essentially bounded portfolio strategy θ such that

$$
\begin{aligned}
c(s^0) + q(s^0)\theta(s^0) &\leq y^i(s^0) + q(s^0)\alpha^i, \\
c(s^t) &\leq y^i(s^t) + z(q, \theta)(s^t) \qquad \forall s^t \neq s^0,.
\end{aligned}
\tag{66}
$$

We have

Lemma A.2. *Let P be a countably additive pricing functional and $q \geq 0$ a market-completing system of security prices with zero price bubbles, i.e., $\sigma_q = 0$. If*

$$
\pi_q(s^t) = p(s^t) \qquad \forall s^t,
\tag{67}
$$

and there exists an essentially bounded portfolio strategy η such that

$$
-\frac{1}{\pi_q(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau) y^i(s^t) \geq q(s^t)\eta(s^t), \qquad \forall s^t,
\tag{68}
$$

then

$$
B_b(q; y^i, \alpha^i) = B_{AD}(P; \omega^i)
\tag{69}
$$

Proof: Let $c \in B_b(q; y^i, \alpha^i)$. As in the proof of Lemma A.1, budget constraint (66) implies (45). Taking limits in (45) as τ goes to infinity, we obtain

$$
\sum_{s^t \in \mathcal{E}} \pi_q(s^t) c(s^t) + \liminf_{\tau \to \infty} \sum_{s^\tau \in \mathcal{F}_\tau} \pi_q(s^\tau) q(s^\tau)\theta(s^\tau) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t) + q(s^0)\alpha^i.
\tag{70}
$$

Note that (68) for s^0 implies that

$$
\sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t) \leq y^i(s^0) - q(s^0)\eta(s^0)
\tag{71}
$$

which shows that $\sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t)$ is finite. We claim that

$$
\liminf_{\tau \to \infty} \sum_{s^\tau \in \mathcal{F}_\tau} \pi_q(s^\tau) q(s^\tau)\theta(s^\tau) \geq 0.
\tag{72}
$$

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Since θ is essentially bounded, it follows that

$$\sum_{s^\tau \in \mathcal{F}_\tau} \pi_q(s^\tau) q(s^\tau) \theta(s^\tau) \geq \sum_{s^\tau \in \mathcal{F}_\tau} \pi_q(s^\tau) q(s^\tau) b(s^\tau) \qquad \forall \tau, \tag{73}$$

for some bounded portfolio strategy b. Since $\sigma_{qj}(s^0) = 0$, (9) implies that the limit, as τ goes to infinity, of the right-hand side of (73) is positive.

Inequalities (72) and (70) imply (48), that is

$$\sum_{s^t \in \mathcal{E}} \pi_q(s^t) c(s^t) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t) y^i(s^t) + q(s^0) \alpha^i, \tag{74}$$

Using the same arguments as in the proof of Lemma A.1 we can rewrite (74) as

$$P(c) \leq P(\omega^i). \tag{75}$$

Thus $c \in B_{AD}(P; \omega^i)$.

Suppose now that $c \in B_{AD}(P; \omega^i)$. In the proof of Lemma A.1 we constructed a portfolio strategy that generates c at security prices q and satisfies the wealth constraint. That is,

$$q(s^t) \theta(s^t) \geq -\frac{1}{\pi_q(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} \pi_q(s^\tau) y^i(s^\tau) \qquad \forall s^t. \tag{76}$$

Using (68) we obtain
$$q(s^t) \theta(s^t) \geq q(s^t) \eta(s^t) \qquad \forall s^t, \tag{77}$$

which implies that θ is essentially bounded. Consequently, $c \in B_b(q; y^i, \alpha^i)$. □

Proof of Theorem 6.2: Lemma A.2 implies that consumption plan c^i is optimal for each i in sequential markets under the constraint of essentially bounded portfolio strategies. The proof that portfolio strategies that generate the optimal consumption plans clear security markets is the same as in Theorem 5.1. □

Proof of Theorem 6.3: It follows immediately from Lemma A.2. □

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