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Takushi Kurozumi and Willem Van Zandweghe

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Takushi Kurozumi† Willem Van Zandweghe‡

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Abstract

A pitfall of expectational stability (E-stability) analysis can arise in models with multi-period expectations: if an auxiliary variable is introduced as substitute for an expectational endogenous variable in such a model, this shrinks the region of the model parameters that guarantee E-stability of a fundamental rational expectations equilibrium. Moreover, in the model representation with no auxiliary variable, the same E-stability region as in that with the auxiliary variable is obtained if economic agents are assumed to make multiple forecasts in an inconsistent manner. Therefore, we argue that the introduction of an auxiliary variable as substitute for an expectational endogenous variable in models with multi-period expectations can induce misleading implications that are biased toward E-instability.

JEL Classification: C62

Keywords: Expectational stability; Model representation; Auxiliary endogenous variable

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†Bank of Japan, 2-1-1 Nihonbashi Hongokucho, Chuo-ku, Tokyo 103-8660, Japan. Tel.: +81 3 3279 1111; fax: +81 3 3510 1265. E-mail address: takushi.kurozumi@boj.or.jp

‡Federal Reserve Bank of Kansas City, 1 Memorial Drive, Kansas City, MO 64198, USA. Tel.: +1 816 881 2766; fax: +1 816 881 2199. E-mail address: willem.vanzandweghe@kc.frb.org

1 Introduction

Learning has been analyzed extensively in modern macroeconomics. In particular, adaptive learning has received much attention since the seminal work by Evans and Honkapohja (1999, 2001). Under this learning, economic agents are supposed to form expectations by estimating and updating their forecasting models in real time. In relation to rational expectations (RE), expectational stability (E-stability) of RE equilibrium has been investigated in a number of macroeconomic areas, including monetary policy analysis. For the RE equilibrium in question, E-stability examines whether an associated equilibrium in which economic agents form expectations under adaptive learning reaches over time that RE equilibrium. E-stability is therefore one of the most important concepts of stability to assess equilibrium under non-RE or learning.

This paper shows a pitfall of E-stability analysis. The pitfall can arise in models with multi-period expectations: if an auxiliary variable is introduced as substitute for an expectational endogenous variable in such a model, this shrinks the region of the model parameters that guarantee E-stability of a fundamental RE equilibrium.[1] This pitfall is demonstrated in E-stability analysis of a simple univariate model with two-period expectations and a dynamic stochastic general equilibrium (DSGE) model with trend inflation that has been widely used in the recent literature.[2] In the univariate model, if an auxiliary variable is incorporated as substitute for a two-period-ahead forcast, this shrinks the region of the model parameters that ensure E-stability of a fundamental RE equilibrium. As for the DSGE model, the recent literature employs two representations of the log-linearized model. One representation contains a two-period-ahead inflation forecast as well as a one-period-ahead one in a generalized New Keynesian Phillips curve. The other representation introduces an auxiliary variable as substitute for the two-period-ahead forecast in the curve. Under plausible calibrations of parameters in the DSGE model, (local) determinacy of RE equilibrium is always identical between the two representations, but E-stability of a fundamental RE equilibrium is more likely in the representation with no auxiliary variable. In particular, the interest rate rule with the inflation and

[1] The term "fundamental RE equilibrium" refers to Evans and Honkapohja (2001)'s minimal state variable (MSV) solutions to linear RE models to distinguish them from McCallum (1983)'s original MSV solution.

[2] See, e.g., Ascari, Castelnuovo, and Rossi (2011), Ascari, Florio, and Gobbi (2014), Ascari and Ropele (2007, 2009), Cogley and Sbordone (2008), Coibion and Gorodnichenko (2011), Kobayashi and Muto (2013), Kurozumi (2013, 2014), and Kurozumi and Van Zandweghe (2012b, 2013). Ascari and Sbordone (2014) review this strand of literature.

output coefficients estimated by Taylor (1993) ensures E-stability in the representation with no auxiliary variable, whereas in that with the auxiliary variable it induces no E-stable fundamental RE equilibrium when trend inflation is high. This result demonstrates that the different model representations can cause opposite implications of E-stability for monetary policy.

Which model representation is appropriate for E-stability analysis, that with or without auxiliary variables? In both the univariate model and the calibrated versions of the DSGE model, this paper shows that the same E-stability region as in the representation with the auxiliary variable is obtained in that with no such variable if economic agents are assumed to make multiple forecasts in an inconsistent manner. The absence of the consistency in the forecasts makes it more difficult for agents to learn the RE and therefore E-stability is less likely. Because such inconsistent forecasts are problematic, we argue that the introduction of an auxiliary variable as substitute for an expectational endogenous variable in models with multi-period expectations can induce misleading implications that are biased toward E-instability.

The remainder of the paper proceeds as follows. Section 2 outlines a pitfall of E-stability analysis using a simple univariate model. Section 3 presents a DSGE model with trend inflation and two representations of the log-linearized model, and analyzes E-stability of a fundamental RE equilibrium. Section 4 concludes.

2 E-stability Analysis of a Simple Univariate Model

This section outlines the pitfall of E-stability analysis using a simple univariate model with two-period expectations.

The model relates an endogenous variable x_t to its one- and two-period-ahead forecasts $\hat{E}_t x_{t+1}$, $\hat{E}_t x_{t+2}$ and an exogenous variable u_t according to

$$x_t = \beta_1 \hat{E}_t x_{t+1} + \beta_2 \hat{E}_t x_{t+2} + u_t, \qquad (1)$$

where $\hat{E}_t$ is a possibly non-RE operator and u_t is a white noise. Note that by the Cohn-Schur criterion (see, e.g., LaSalle, 1986) the condition for determinacy of equilibrium in this model consists of $|\beta_2| < 1$ and $|\beta_1| < 1 - \beta_2$. The fundamental RE equilibrium is given by

$$x_t = \bar{c}_x + \bar{\Gamma}_x u_t, \quad (\bar{c}_x, \bar{\Gamma}_x) = (0, 1).$$

Following Section 10.3 of Evans and Honkapohja (2001), E-stability of the fundamental RE equilibrium is analyzed. Corresponding to this equilibrium, economic agents are assumed to be endowed with the perceived law of motion (PLM) of x_t given by $x_t = c_x + \Gamma_x u_t$. Then, the model implies that the actual law of motion (ALM) of x_t is given by $x_t = (\beta_1 + \beta_2)c_x + u_t$. The mapping T_x from the PLM to the ALM can thus be defined as $T_x(c_x, \Gamma_x) = ((\beta_1+\beta_2)c_x, 1)$. For the fundamental RE equilibrium $(\bar{c}_x, \bar{\Gamma}_x)$ to be E-stable, the differential equation $\frac{d}{d\tau}(c_x, \Gamma_x) = T(c_x, \Gamma_x) - (c_x, \Gamma_x)$, where τ denotes a notional time, must have local asymptotic stability at the equilibrium, that is, $D_c T_x(\bar{c}_x, \bar{\Gamma}_x) = \beta_1 + \beta_2$ and $D_\Gamma T_x(\bar{c}_x, \bar{\Gamma}_x) = 0$ are less than unity. Hence, the E-stability condition is $\beta_1 + \beta_2 < 1$.[3]

Next, we consider an alternative representation of the model (1) using the auxiliary variable $\psi_t = \hat{E}_t x_{t+1}$. The representation is given by

$$x_{1,t} = A\hat{E}_t x_{1,t+1} + Bu_t, \quad x_{1,t} = \begin{bmatrix} x_t \\ \psi_t \end{bmatrix}, \quad A = \begin{bmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2}$$

Note that the determinacy condition for this representation is the same as that for representation (1). The fundamental RE equilibrium in representation (2) is given by

$$x_{1,t} = \bar{c}_{x1} + \bar{\Gamma}_{x1} u_t, \quad (\bar{c}_{x1}, \bar{\Gamma}_{x1}) = (0_{2\times 1}, B), \tag{3}$$

so that the fundamental RE equilibrium is the same as above. As is similar to E-stability analysis of representation (1), assuming that the PLM of x_{1t} is given by $x_{1,t} = c_{x1} + \Gamma_{x1} u_t$ corresponding to the fundamental RE equilibrium (3), the representation (2) yields the ALM of $x_{1,t}$ given by $x_{1,t} = Ac_{x1} + Bu_t$, and thus the mapping T_{x1} from the PLM to the ALM is defined as $T_{x1}(c_{x1}, \Gamma_{x1}) = (Ac_{x1}, B)$. Therefore, it follows that the fundamental RE equilibrium (3) is E-stable if and only if all eigenvalues of the matrices $D_c T_{x1}(\bar{c}_{x1}, \bar{\Gamma}_{x1}) = A$ and $D_\Gamma T_{x1}(\bar{c}_{x1}, \bar{\Gamma}_{x1}) = 0_{2\times 2}$ have real parts less than unity. By the Routh-Hurwitz theorem (see, e.g., Samuelson, 1947), the E-stability condition for representation (2) consists of $\beta_1 < 2$ and $\beta_1 + \beta_2 < 1$.[4]

The two E-stability conditions for the two representations (1) and (2) imply that the region of the model parameters ensuring E-stability of the fundamental RE equilibrium is smaller in the representation with the auxiliary variable. Moreover, the same E-stability region as in the representation with the auxiliary variable is obtained in that with no such variable if economic agents are assumed to make forecasts in the following inconsistent manner. The model (1) can

[3] This E-stability condition is implied by the determinacy condition $|\beta_1| < 1 - \beta_2$, so that a determinate RE equilibrium is always E-stable in the model.

[4] This E-stability condition is also implied by the determinacy condition presented above.

be rewritten as

$$\tilde{x}_t = A\hat{E}_t\tilde{x}_{t+1} + Bu_t, \quad \tilde{x}_t = \begin{bmatrix} x_t \\ \hat{E}_t x_{t+1} \end{bmatrix}, \tag{4}$$

which is the same as representation (2) when the expectational variable $\hat{E}_t x_{t+1}$ is substituted for the auxiliary variable ψ_t in (2). Therefore, if economic agents are assumed to be endowed with the PLM of $\tilde{x}_t$ given by $\tilde{x}_t = c_{x1} + \Gamma_{x1} u_t$ corresponding to the fundamental RE equilibrium $\tilde{x}_t = \bar{c}_{x1} + \bar{\Gamma}_{x1} u_t$, then the E-stability condition for representation (4) is the same as that for representation (2). Here, the agents' forecasting is inconsistent in that in the PLM of $\tilde{x}_t$ the one-period-ahead forecast is $\hat{E}_t x_{t+1} = c_{x1,2} + \Gamma_{x1,2} u_t$, where $c_{x1,2}$ and $\Gamma_{x1,2}$ are the second elements of the vectors c_{x1} and Γ_{x1}, while in the forecast $\hat{E}_t\tilde{x}_{t+1}$ it is $\hat{E}_t x_{t+1} = c_{x1,1}$, where $c_{x1,1}$ is the first element of c_{x1}.

The remainder of the paper shows that a similar issue as presented in this section can arise in a DSGE model that has been widely used in the recent literature, and that the introduction of an auxiliary variable as substitute for an expectational endogenous variable in the model can induce misleading implications for monetary policy, which are biased toward E-instability.

3 E-stability Analysis of a DSGE Model with Trend Inflation

This section presents E-stability analysis of a DSGE model with trend inflation. The model is a Calvo (1983)-style sticky price model with the Taylor (1993) rule and no price indexation to past or trend inflation, based on Ascari and Ropele (2009) and Kurozumi (2014). The absence of price indexation is consistent not only with micro evidence that each period a fraction of prices is kept unchanged under a positive trend inflation,[5] but also with recent empirical macroeconomic studies, such as Ascari, Castelnuovo, and Rossi (2011) and Cogley and Sbordone (2008).

In the model economy, there are a representative household, two types of firms, and a monetary policymaker. This section describes each agent's behavior in turn.

[5] For recent micro evidence on price adjustment, see, e.g., Bils and Klenow (2004), Kehoe and Midrigan (2012), Klenow and Kryvtsov (2008), Klenow and Malin (2010), and Nakamura and Steinsson (2008).

3.1 Representative household

The representative household consumes final goods c_t, purchases one-period riskless bonds B_t, and supplies labor n_t to maximize the utility function $E_0 \sum_{t=0}^{\infty} \beta^t \{\log(c_t) - n_t^{1+\sigma_n}/(1+\sigma_n)\exp(\tilde{u}_t)\}$ subject to the budget constraint $P_t c_t + B_t = P_t w_t n_t + i_{t-1} B_{t-1} + T_t$, where E_t is the RE operator conditional on information available in period t, $\beta \in (0,1)$ is the subjective discount factor, $\sigma_n \geq 0$ is the inverse of the elasticity of labor supply, $\tilde{u}_t$ is a shock to labor disutility relative to contemporaneous consumption utility, P_t is the price of final goods, w_t is the real wage, i_t is the gross interest rate on bonds, which is assumed to equal the monetary policy rate, and T_t consists of lump-sum public transfers and firm profits. The shock $\tilde{u}_t$ is assumed to follow a stationary first-order autoregressive process with the persistence parameter $\rho \in [0,1)$.

The first-order conditions for utility maximization with respect to consumption, labor supply, and bond holdings are

$$\lambda_t = \frac{1}{c_t}, \tag{5}$$

$$w_t = \frac{n_t^{\sigma_n} \exp(\tilde{u}_t)}{\lambda_t}, \tag{6}$$

$$1 = E_t \frac{\beta \lambda_{t+1}}{\lambda_t} \frac{i_t}{\pi_{t+1}}, \tag{7}$$

where λ_t is the marginal utility of consumption and $\pi_t = P_t/P_{t-1}$ is the gross inflation rate of the final-good price. Throughout the section, the trend inflation rate is assumed to be nonnegative, i.e., $\pi \geq 1$.

3.2 Firms

There are a representative final-good firm and a continuum of intermediate-good firms $j \in [0,1]$.

The final-good firm produces homogeneous goods y_t by choosing a combination of differentiated intermediate inputs $\{y_t(j)\}$ to maximize profit $P_t y_t - \int_0^1 P_t(j) y_t(j)\, dj$ subject to the CES production technology $y_t = \{\int_0^1 (y_t(j))^{(\theta-1)/\theta} dj\}^{\theta/(\theta-1)}$, where $P_t(j)$ is the price of intermediate good j and $\theta > 1$ is the price elasticity of demand for each intermediate good.

The first-order condition for profit maximization yields the final-good firm's demand curve for intermediate good j

$$y_t(j) = y_t \left(\frac{P_t(j)}{P_t}\right)^{-\theta}, \tag{8}$$

while perfect competition in the final-good market leads to

$$P_t = \left\{ \int_0^1 (P_t(j))^{1-\theta} \, dj \right\}^{\frac{1}{1-\theta}}. \tag{9}$$

Each intermediate-good firm j produces one kind of differentiated good $y_t(j)$ according to the production function that is linear in labor input $n_t(j)$,

$$y_t(j) = n_t(j). \tag{10}$$

The first-order condition for firm j's minimization of production cost shows that real marginal cost is identical among intermediate-good firms and is given by

$$mc_t = mc_t(j) = w_t. \tag{11}$$

In the face of the final-good firm's demand curve (8) and the marginal cost (11), intermediate-good firms set prices of their products on a staggered basis as in Calvo (1983). Each period a fraction $\alpha \in (0, 1)$ of firms keeps the previous-period prices unchanged, while the remaining fraction $1 - \alpha$ of firms sets the price $P_t(j)$ to maximize the associated profit function $E_t \sum_{\tau=0}^{\infty} \alpha^\tau q_{t,t+\tau}(P_t(j)/P_{t+\tau} - mc_{t+\tau})y_{t+\tau|t}(i)$ subect to the demand curve $y_{t+\tau|t}(i) = y_{t+\tau}(P_t(j)/P_{t+\tau})^{-\theta}$, where $q_{t,t+\tau}$ is the stochastic discount factor between period t and period $t+\tau$. For this profit function to be well-defined, it is assumed throughout this section that the condition $\alpha\pi^\theta < 1$ is satisfied.

Using the equilibrium condition $q_{t,t+\tau} = \beta^\tau \lambda_{t+\tau}/\lambda_t$, the first-order condition for intermediate-good firms that reset prices in period t becomes

$$E_t \sum_{\tau=0}^{\infty} (\alpha\beta)^\tau \frac{\lambda_{t+\tau}}{\lambda_t} y_{t+\tau} \left(p_t^* \prod_{k=1}^{\tau} \frac{1}{\pi_{t+k}} \right)^{-\theta} \left(p_t^* \prod_{k=1}^{\tau} \frac{1}{\pi_{t+k}} - \frac{\theta}{\theta - 1} mc_{t+\tau} \right) = 0, \tag{12}$$

where $p_t^* = P_t^*/P_t$ and P_t^* is the price reset by the firms.

The final-good market clearing condition is given by

$$y_t = c_t. \tag{13}$$

The labor market clearing condition, along with the final-good firm's demand curve (8) and intermediate-good firms' production function (10), yields

$$n_t = \int_0^1 n_t(j) dj = y_t s_t, \tag{14}$$

where s_t represents price distortion given by $s_t = \int_0^1 (P_t(j)/P_t)^{-\theta} dj$. Under the staggered price setting, this price distortion equation can be reduced to

$$s_t = (1 - \alpha) (p_t^*)^{-\theta} + \alpha \pi_t^{\theta} s_{t-1}. \tag{15}$$

The final-good price equation (9) can also be reduced to

$$1 = (1 - \alpha) (p_t^*)^{1-\theta} + \alpha \pi_t^{\theta-1}. \tag{16}$$

3.3 Monetary policy

The monetary policymaker follows the Taylor (1993) rule

$$\log i_t = \log i + \phi_\pi (\log \pi_t - \log \pi) + \phi_y (\log y_t - \log y), \tag{17}$$

where i is the steady-state gross rate of monetary policy and $\phi_\pi, \phi_y \geq 0$ are the degrees of responses of the policy rate ($\log i_t$) to deviations of the inflation rate ($\log \pi_t$) and output ($\log y_t$) from their steady-state values ($\log \pi$, $\log y$).

3.4 Two representations of the log-linearized model

The equilibrium conditions are given by (5)–(7), (11)–(16), and (17). Log-linearizing these conditions and rearranging the resulting equations leads to the following two representations of the log-linearized model.

The representation (I) is given by

$$\hat{y}_t = E_t \hat{y}_{t+1} - \left(\hat{i}_t - E_t \hat{\pi}_{t+1} \right), \tag{18}$$

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t, \tag{19}$$

$$\hat{mc}_t = (1 + \sigma_n) \hat{y}_t + \sigma_n \hat{s}_t + \tilde{u}_t, \tag{20}$$

$$\hat{s}_t = \frac{\theta \alpha \pi^{\theta-1} (\pi - 1)}{1 - \alpha \pi^{\theta-1}} \hat{\pi}_t + \alpha \pi^{\theta} \hat{s}_{t-1}, \tag{21}$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{(1 - \alpha \pi^{\theta-1})(1 - \alpha \beta \pi^{\theta})}{\alpha \pi^{\theta-1}} \hat{mc}_t + \psi_t, \tag{22}$$

$$\psi_t = \alpha \beta \pi^{\theta} E_t \psi_{t+1} + \beta (\pi - 1)(1 - \alpha \pi^{\theta-1}) \left\{ \theta E_t \hat{\pi}_{t+1} + (1 - \alpha \beta \pi^{\theta}) E_t \hat{mc}_{t+1} \right\}, \tag{23}$$

where hatted variables denote log-deviations from steady-state values and ψ_t is an auxiliary variable. The representation of this sort—the one with the auxiliary variable—is used in recent studies, such as Ascari, Castelnuovo, and Rossi (2011), Ascari and Ropele (2009), Ascari, Florio, and Gobbi (2014), Kobayashi and Muto (2013), and Kurozumi and Van Zandweghe (2013).

The representation (II) consists of (18)–(21), and

$$\hat{\pi}_t - \alpha\beta\pi^{\theta} E_t\hat{\pi}_{t+1} = \beta\Big(E_t\hat{\pi}_{t+1} - \alpha\beta\pi^{\theta} E_t\hat{\pi}_{t+2}\Big) + \beta\theta(\pi-1)(1-\alpha\pi^{\theta-1})E_t\hat{\pi}_{t+1} + \frac{(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})}{\alpha\pi^{\theta-1}}\Big(\hat{mc}_t - \alpha\beta\pi^{\theta-1}E_t\hat{mc}_{t+1}\Big). \tag{24}$$

The representation of this sort—the one with no auxiliary variable—is used in Kurozumi (2013, 2014) and Kurozumi and Van Zandweghe (2012b). Compared with representation (I), this representation contains two-period-ahead inflation forecast $E_t\hat{\pi}_{t+2}$ instead of the auxiliary variable ψ_t.

Note that each of (22) and (24) is a generalized New Keynesian Phillips curve. This is because under the zero trend inflation rate (i.e., $\pi = 1$), (23) implies $\psi_t = 0$ and hence (22) becomes

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\hat{mc}_t. \tag{25}$$

On the other hand, (24) becomes

$$\hat{\pi}_t - \alpha\beta E_t\hat{\pi}_{t+1} = \beta\left(E_t\hat{\pi}_{t+1} - \alpha\beta E_t\hat{\pi}_{t+2}\right) + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\left(\hat{mc}_t - \alpha\beta E_t\hat{mc}_{t+1}\right),$$

which can be reduced to (25) because $0 < \alpha\beta < 1$.

The generalized New Keynesian Phillips curve depends on price distortion $\hat{s}_t$ through the real marginal cost (20) as long as the elasticity of labor supply is finite (i.e., $\sigma_n > 0$). Then, under a positive trend inflation rate (i.e., $\pi > 1$), the persistence of price distortion described in (21) generates endogenously persistent inflation dynamics. As emphasized by Kurozumi (2014), this endogenous persistence of inflation dynamics through price distortion helps agents learn the RE and as a consequence, a fundamental RE equilibrium is likely to be E-stable in representation (II) under a plausible calibration of the model parameters. Therefore, a positive trend inflation rate and a finite elasticity of labor supply are both key to E-stability of a fundamental RE equilibrium in representation (II).

With the two representations of the log-linearized model presented above, the next sections examine E-stability of a fundamental RE equilibrium.

3.5 E-stability condition

Following the literature on learning in macroeconomics (e.g., Evans and Honkapohja, 2001), the present paper adopts the so-called "Euler equation" approach suggested by Honkapohja et al. (2011): the RE operator E_t is replaced with a possibly non-RE operator $\hat{E}_t$ in each of the two representations (I) and (II).

By using (20) to eliminate the real marginal cost terms, representation (I) can be rewritten as

$$z_{1,t} = A_1 \hat{E}_t z_{1,t+1} + C_1 \hat{s}_{t-1} + D_1 u_t, \tag{26}$$

where $z_{1,t} = [\hat{\pi}_t \; \hat{y}_t \; \hat{s}_t \; \psi_t]'$ and the coefficient matrix A_1 and the coefficient vector C_1 are given in Appendix A.[6] In this system, a fundamental RE equilibrium is given by

$$z_{1,t} = \bar{c}_1 + \bar{\Phi}_1 \hat{s}_{t-1} + \bar{\Gamma}_1 u_t, \tag{27}$$

where the coefficient vectors $\bar{c}_1$, $\bar{\Phi}_1$, $\bar{\Gamma}_1$ are determined by

$$\bar{c}_1 = 0_{4\times 1}, \quad A_1 \bar{\Phi}_1 [0\;0\;1\;0] \bar{\Phi}_1 = \bar{\Phi}_1 - C_1, \quad \bar{\Gamma}_1 = \{I - A_1(\rho I + \bar{\Phi}_1 [0\;0\;1\;0])\}^{-1} D_1.$$

Note that $\bar{\Gamma}_1$ is uniquely determined given a $\bar{\Phi}_1$, whereas $\bar{\Phi}_1$ is not generally uniquely determined, which causes multiplicity of the fundamental RE equilibrium (27).

Following Section 10.5 of Evans and Honkapohja (2001), E-stability of the fundamental RE equilibrium (27) is investigated.[7] Corresponding to this equilibrium, all agents are assumed to be endowed with the PLM of $z_{1,t}$

$$z_{1,t} = c_1 + \Phi_1 \hat{s}_{t-1} + \Gamma_1 u_t. \tag{28}$$

[6]The form of the coefficient vector D_1 is omitted, since it is not needed in what follows.

[7]The system (26) contains the predetermined endogenous variable $\hat{s}_{t-1}$ and thus it is possible to consider two learning environments, which are studied respectively in Sections 10.3 and 10.5 of Evans and Honkapohja (2001). One environment allows agents to use current endogenous variables in expectation formation, whereas the other does not. The present paper considers only the latter environment as in Bullard and Mitra (2002), Kurozumi (2006, 2014), and Kurozumi and Van Zandweghe (2008, 2012a). This is because the former induces a problem with simultaneous determination of expectations and current endogenous variables, which is critical to equilibrium under non-RE as indicated by Evans and Honkapohja (2001) and Bullard and Mitra (2002).

Using a forecast from this PLM and the relation $\hat{s}_t = [0\ 0\ 1\ 0]z_{1,t}$ to substitute $\hat{E}_t z_{1,t+1}$ out of (26) leads to the ALM of $z_{1,t}$

$$z_{1,t} = A_1(I + \Phi_1[0\ 0\ 1\ 0])c_1 + (A_1\Phi_1[0\ 0\ 1\ 0]\Phi_1 + C_1)\hat{s}_{t-1} + \{A_1(\rho I + \Phi_1[0\ 0\ 1\ 0])\Gamma_1 + D_1\}u_t. \tag{29}$$

The mapping T_1 from the PLM (28) to the ALM (29) can thus be defined as

$$T_1 \begin{pmatrix} c_1 \\ \Phi_1 \\ \Gamma_1 \end{pmatrix}' = \begin{pmatrix} A_1(I + \Phi_1[0\ 0\ 1\ 0])c_1 \\ A_1\Phi_1[0\ 0\ 1\ 0]\Phi_1 + C_1 \\ A_1(\rho I + \Phi_1[0\ 0\ 1\ 0])\Gamma_1 + D_1 \end{pmatrix}'.$$

Therefore, a fundamental RE equilibrium $(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1)$ is E-stable if and only if all eigenvalues of the three matrices $DT_{1,c}(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1)$, $DT_{1,\Phi}(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1)$, $DT_{1,\Gamma}(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1)$ have real parts less than unity. Since

$$\begin{aligned} DT_{1,c}(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1) &= A_1(I + \bar{\Phi}_1[0\ 0\ 1\ 0]), \\ DT_{1,\Phi}(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1) &= A_1(\bar{\Phi}_{1,3}I + \bar{\Phi}_1[0\ 0\ 1\ 0]), \\ DT_{1,\Gamma}(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1) &= A_1(\rho I + \bar{\Phi}_1[0\ 0\ 1\ 0]), \end{aligned}$$

where $\bar{\Phi}_{1,3}$ is the third element of the RE equilibrium coefficient vector $\bar{\Phi}_1$, the E-stability condition for a fundamental RE equilibrium $(\bar{c}_1, \bar{\Phi}_1, \bar{\Gamma}_1)$ is that all eigenvalues of the three matrices $A_1(\varphi I + \bar{\Phi}_1[0\ 0\ 0\ 1])$, $\varphi \in \{1,\ \rho,\ \bar{\Phi}_{1,3}\}$ have real parts less than unity.

As for representation (II), it can be written as

$$z_{2,t} = A_2\hat{E}_t z_{2,t+1} + B_2[1\ 0\ 0]\hat{E}_t z_{2,t+2} + C_2\hat{s}_{t-1} + D_2 u_t, \tag{30}$$

where $z_{2,t} = [\hat{\pi}_t\ \hat{y}_t\ \hat{s}_t]'$ and the coefficient matrix A_2 and the coefficient vectors B_2, C_2 are given in Appendix A.[8] In this system, fundamental RE equilibrium is given by

$$z_{2,t} = \bar{c}_2 + \bar{\Phi}_2\hat{s}_{t-1} + \bar{\Gamma}_2 u_t, \tag{31}$$

where the coefficient vectors $\bar{c}_2$, $\bar{\Phi}_2$, $\bar{\Gamma}_2$ are determined by

$$\begin{aligned} &\bar{c}_2 = 0_{3\times 1}, \quad B_2[1\ 0\ 0]\bar{\Phi}_2[0\ 0\ 1]\bar{\Phi}_2[0\ 0\ 1]\bar{\Phi}_2 + A_2\bar{\Phi}_2[0\ 0\ 1]\bar{\Phi}_2 = \bar{\Phi}_2 - C_2, \\ &\bar{\Gamma}_2 = \{I - A_2(\rho I + \bar{\Phi}_2[0\ 0\ 1]) - B_2[1\ 0\ 0](\rho^2 I + \rho\bar{\Phi}_2[0\ 0\ 1] + \bar{\Phi}_2[0\ 0\ 1]\bar{\Phi}_2[0\ 0\ 1])\}^{-1}D_2. \end{aligned}$$

[8]The form of the coefficient vector D_2 is omitted, since it is not needed in what follows.

Corresponding to this equilibrium, all agents are assumed to be endowed with the PLM of $z_{2,t}$

$$z_{2,t} = c_2 + \Phi_2 \hat{s}_{t-1} + \Gamma_2 u_t. \tag{32}$$

Using a forecast from this PLM and the relation $\hat{s}_t = [0\ 0\ 1] z_{2,t}$ to substitute $\hat{E}_t z_{2,t+1}$ and $\hat{E}_t z_{2,t+2}$ out of the system (30) leads to the ALM of $z_{2,t}$

$$\begin{aligned} z_{2,t} = &\{A_2(I + \Phi_2[0\ 0\ 1]) + B_2[1\ 0\ 0](I + \Phi_2[0\ 0\ 1] + \Phi_2[0\ 0\ 1]\Phi_2[0\ 0\ 1])\}c_2 \\ &+ \{(A_2 + B_2[1\ 0\ 0]\Phi_2[0\ 0\ 1])\Phi_2[0\ 0\ 1]\Phi_2 + C_2\}\hat{s}_{t-1} \\ &+ \{A_2(\rho I + \Phi_2[0\ 0\ 1])\Gamma_2 + B_2[1\ 0\ 0](\rho^2 I + \rho\Phi_2[0\ 0\ 1] + \Phi_2[0\ 0\ 1]\Phi_2[0\ 0\ 1])\Gamma_2 + D_2\}u_t. \end{aligned} \tag{33}$$

Thus, the mapping T_2 from the PLM (32) to the ALM (33) can be defined as

$$T_2 \begin{pmatrix} c_2 \\ \Phi_2 \\ \Gamma_2 \end{pmatrix}' = \begin{pmatrix} \{A_2(I + \Phi_2[0\ 0\ 1]) + B_2[1\ 0\ 0](I + \Phi_2[0\ 0\ 1] + \Phi_2[0\ 0\ 1]\Phi_2[0\ 0\ 1])\}c_2 \\ (A_2 + B_2[1\ 0\ 0]\Phi_2[0\ 0\ 1])\Phi_2[0\ 0\ 1]\Phi_2 + C_2 \\ A_2(\rho I + \Phi_2[0\ 0\ 1])\Gamma_2 + B_2[1\ 0\ 0](\rho^2 I + \rho\Phi_2[0\ 0\ 1] + \Phi_2[0\ 0\ 1]\Phi_2[0\ 0\ 1])\Gamma_2 + D_2 \end{pmatrix}'. \tag{34}$$

Consequently, the E-stability condition for a fundamental RE equilibrium $(\bar{c}_2, \bar{\Phi}_2, \bar{\Gamma}_2)$ is that all eigenvalues of the following three matrices have real parts less than unity.

$$\begin{aligned} DT_{2,c}(\bar{c}_2, \bar{\Phi}_2, \bar{\Gamma}_2) &= A_2(I + \bar{\Phi}_2[0\ 0\ 1]) + B_2[1\ 0\ 0](I + \bar{\Phi}_2[0\ 0\ 1] + \bar{\Phi}_2[0\ 0\ 1]\bar{\Phi}_2[0\ 0\ 1]), \\ DT_{2,\Phi}(\bar{c}_2, \bar{\Phi}_2, \bar{\Gamma}_2) &= A_2(\bar{\Phi}_{2,3} I + \bar{\Phi}_2[0\ 0\ 1]) + B_2[1\ 0\ 0]\{(\bar{\Phi}_{2,3})^2 I + \bar{\Phi}_{2,3}\bar{\Phi}_2[0\ 0\ 1] + \bar{\Phi}_2[0\ 0\ 1]\bar{\Phi}_2[0\ 0\ 1]\}, \\ DT_{2,\Gamma}(\bar{c}_2, \bar{\Phi}_2, \bar{\Gamma}_2) &= A_2(\rho I + \bar{\Phi}_2[0\ 0\ 1]) + B_2[1\ 0\ 0](\rho^2 I + \rho\bar{\Phi}_2[0\ 0\ 1] + \bar{\Phi}_2[0\ 0\ 1]\bar{\Phi}_2[0\ 0\ 1]), \end{aligned}$$

where $\bar{\Phi}_{2,3}$ is the third element of the RE equilibrium coefficient vector $\bar{\Phi}_2$.

For each of the two representations (I) and (II), E-stability of the fundamental RE equilibrium (27) and (31) is numerically investigated, since it seems impossible to analytically solve the matrix equations for the RE equilibrium coefficient vectors $\bar{\Phi}_1$ and $\bar{\Phi}_2$. As McCallum (1998) indicates, distinct fundamental RE equilibria are obtained for different orderings of stable generalized eigenvalues of the matrix pencil for each of the systems (26) and (30). Indeed, the calibrations of the model parameters presented below show that in cases of indeterminacy there are two or three distinct fundamental RE equilibrium of the form (27) and (31).

3.6 Calibrations of model parameters

The ensuing analysis uses plausible calibrations of the model parameters to illustrate regions of the parameter space in which E-stability of a fundamental RE equilibrium is guaranteed in each of the two representations (I) and (II). The calibrations for the quarterly model are summarized in Table 1. In line with Ascari and Ropele (2009) and Kurozumi (2014), the present paper sets the subjective discount factor at $\beta = 0.99$, the probability of no price adjustment at $\alpha = 0.75$, and the price elasticity of demand for differentiated intermediate goods at $\theta = 11$. The assumption of $\alpha\pi^{\theta} < 1$ then requires that the annualized trend inflation rate does not exceed 11%. As noted above, a positive trend inflation rate is key to E-stability of a fundamental RE equilibrium in representation (II) and thus its annualized rate is set at two, four, six, and eight percent, i.e., $\pi = 1.005, 1.010, 1.015, 1.020$. The elasticity of labor supply is also key to the E-stability and thus the inverse of the elasticity of labor supply is set at $\sigma_n = 1, 2$.[9] The shock persistence is chosen at $\rho = 0.8$ as in Woodford (2003) and Kurozumi (2014).

3.7 Results of E-stability analysis

This subsection shows that representation (I) induces a smaller region of the model parameters that guarantee E-stability of a fundamental RE equilibrium than representation (II), using the calibrations presented in Table 1. Because the trend inflation rate and the elasticity of labor supply are both key to differences between the two representations as shown below, how the E-stability region of each representation varies with trend inflation is explained in each case of the calibrations of the elasticity.

We begin with the case of the elasticity of labor supply of unity (i.e., $\sigma_n = 1$). Fig. 1 illustrates the regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium in representation (I). This paper focuses on the range of the Taylor rule's coefficients given by $0 \leq \phi_\pi \leq 4.5$ and $0 \leq \phi_y \leq 1.5/4 = 0.375$. Because the estimates of these coefficients by Taylor (1993) are

[9] Hall (2009) surveys the recent empirical literature on the Frisch elasticity of labor supply and concludes that an empirically plausible value of the elasticity is 0.7, which lies within the range implied by the two values of the inverse of the elasticity in our calibrations. The case of an infinite elasticity of labor supply (i.e., $\sigma_n = 0$)—which is not empirically plausible but is analyzed in some theoretical macroeconomic studies—is also presented in Appendix B.

$\phi_\pi = 1.5$ and $\phi_y = 0.5/4 = 0.125$, it is reasonable to set upper bounds on the coefficients at the values three times larger than the estimates. Note that in the region of E-stability at least one E-stable fundamental RE equilibrium is generated, while in the region of E-instability no fundamental RE equilibrium is E-stable. The figure shows that high trend inflation is a serious cause of E-instability of all fundamental RE equilibrium as well as indeterminacy of RE equilibrium. In the case of the annualized trend inflation rate of two percent, the upper-left panel of the figure shows two regions of the Taylor rule's coefficients: one region ensures determinacy and E-stability of RE equilibrium and the other induces indeterminacy of RE equilibrium and E-instability of all fundamental RE equilibrium. The former region is fairly wide and contains the estimates of Taylor (1993). The boundary between the two regions is given by $\phi_\pi + \epsilon_y \phi_y = 1$, where

$$\epsilon_y = \frac{\alpha\pi^{\theta-1}[(1-\beta)(1-\alpha\pi^\theta)(1-\alpha\beta\pi^\theta) - \theta(\pi-1)\{\beta(1-\alpha\pi^{\theta-1})(1-\alpha\pi^\theta) + \sigma_n(1-\alpha\beta\pi^{\theta-1})(1-\alpha\beta\pi^\theta)\}]}{(1+\sigma_n)(1-\alpha\pi^{\theta-1})(1-\alpha\pi^\theta)(1-\alpha\beta\pi^{\theta-1})(1-\alpha\beta\pi^\theta)}.$$

Then, the condition

$$\phi_\pi + \epsilon_y \phi_y > 1 \tag{35}$$

characterizes the region of the Taylor rule's coefficients that ensure determinacy and E-stability.

The condition (35) can be interpreted as the long-run version of the Taylor principle. From the law of motion of price distortion (21), the generalized New Keynesian Phillips curve (22), and the equation for the auxiliary variable (23), each percentage point of permanently higher inflation implies ϵ_y percentage points of permanently higher output. Thus, ϵ_y represents the long-run inflation elasticity of output. Then, $\phi_\pi + \epsilon_y \phi_y$ shows the permanent increase in the interest rate by the Taylor rule (19) in response to each unit permanent increase in inflation. Therefore, the condition (35) suggests that in the long run the interest rate should be raised by more than the increase in inflation. This Taylor principle (35) restricts the size of the output coefficient more severely under higher trend inflation, since the value of the elasticity ϵ_y decreases to become negative and further declines as trend inflation rises. For the annualized trend inflation rate less than a threshold (e.g., 2.3% under the calibration of model parameters), the Taylor principle (35) is the relevant, necessary and sufficient condition for determinacy and E-stability, as is similar to the result of Bullard and Mitra (2002) who study the case of the zero trend inflation rate.

As trend inflation increases beyond the threshold, the region of the Taylor rule's coefficients in which E-stability of a fundamental RE equilibrium is ensured narrows remarkably and the one in which determinacy of RE equilibrium is guaranteed further narrows.[10] In the cases of the annualized trend inflation rate of four, six, and eight percent, the Taylor principle (35) is no longer a sufficient condition for E-stability or determinacy, but remains a necessary condition. Moreover, in the cases of the annualized trend inflation rate of six and eight percent, the Taylor (1993) estimates (i.e., $(\phi_\pi, \phi_y) = (1.5, 0.125)$) induce E-instability of all fundamental RE equilibrium as well as indeterminacy of RE equilibrium.

As for representation (II), Fig. 2 illustrates the regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium. The comparison of this figure with Fig. 1 shows that determinacy of RE equilibrium is identical between the two representations (I) and (II), but E-stability of a fundamental RE equilibrium is more likely in representation (II). For annualized trend inflation rates less than the threshold (e.g., 2.3%), the E-stability region is identical to that in representation (I), as shown in the upper-left panel where the rate is two percent. However, once the rate increases beyond the threshold, the E-stability region differs from that in representation (I). In particular, when the rate rises, the E-stability region narrows much less than that in representation (I), as shown in the cases of the annualized trend inflation rate of four, six, and eight percent.

We emphasize two policy implications generated under representation (II), as they differ importantly from the implications under representation (I). First, the Taylor (1993) estimates (i.e., $(\phi_\pi, \phi_y) = (1.5, 0.125)$) lead to E-stability even in the cases of the annualized trend inflation rate of six and eight percent. Therefore, a fundamental RE equilibrium is likely to be E-stable even under high trend inflation. Second, in these cases the Taylor principle (35) is neither a sufficient condition nor a necessary condition for E-stability.[11] This contrasts with representation (I), where the Taylor principle (35) remains a necessary condition for E-stability even under high trend inflation.

We turn next to the case of the elasticity of labor supply of one half (i.e., $\sigma_n = 2$). Figs. 3 and 4 illustrate the regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium in the representations (I) and (II), respectively. The qualitative properties of the results obtained in the case of $\sigma_n = 1$

[10]The result regarding equilibrium determinacy is in line with that of Ascari and Ropele (2009).

[11]In the upper-right panel of Fig. 2, where the annualized trend inflation rate is four percent, there is a small region of the Taylor rule's coefficients that do not meet the Taylor principle (35), where E-stability is ensured ($1.05 \leq \phi_\pi \leq 1.40$).

still hold, but the difference in the E-stability region between the two representations (I) and (II) is much starker. In particular, a lower elasticity of labor supply makes a fundamental RE equilibrium more likely to be E-stable in representation (II). In terms of policy implications, E-stability is ensured under the Taylor (1993) estimates in representation (II) even at the high rates of trend inflation, as the Taylor principle (35) is neither a necessary nor sufficient condition for E-stability. By contrast, in representation (I), E-instability is induced under the estimates if trend inflation is sufficiently high, since the Taylor principle (35) becomes more likely to be violated.

As noted above, in representation (II), a finite elasticity of labor supply causes price distortion to affect inflation dynamics represented by the generalized New Keynesian Phillips curve (24), and the persistence of price distortion in (21) then generates endogenously persistent inflation dynamics. For the RE equilibrium in question, E-stability examines whether an associated equilibrium in which agents form expectations based on a PLM reaches over time that RE equilibrium. Under such expectation formation (i.e., the presence of lagged price distortion $\hat{s}_{t-1}$ in the PLM (32)), the endogenous persistence of inflation dynamics through price distortion helps agents learn the RE. Consequently, E-stability of a fundamental RE equilibrium is likely.[12] The next subsection explains why this does not hold for representation (I).

3.8 Why does representation (I) induce a smaller E-stability region than representation (II)?

This subsection addresses the question of why representation (I) induces a smaller E-stability region than representation (II) as illustrated above. Specifically, the subsection demonstrates that in representation (II), if economic agents make inflation forecasts in an inconsistent manner, the resulting E-stability region becomes identical with that in representation (I).

The system of representation (II) given by (30) can be rewritten as

$$z_{3,t} = A_3 \hat{E}_t z_{3,t+1} + C_3 \hat{s}_{t-1} + D_3 u_t, \tag{36}$$

where $z_{3,t} = [z'_{2,t} \; \hat{E}_t \hat{\pi}_{t+1}]' (= [\hat{\pi}_t \; \hat{y}_t \; \hat{s}_t \; \hat{E}_t \hat{\pi}_{t+1}]')$ and the coefficient matrix A_3 and the coefficient

[12] Consistently, in the case of an infinite elasticity of labor supply (i.e., $\sigma_n = 0$), where price distortion never affects inflation dynamics, high trend inflation is a serious cause of E-instability of the fundamental RE equilibrium as well as indeterminacy of RE equilibrium. See Appendix B for the analysis of this case.

vectors C_3, D_3 are given by

$$A_3 = \begin{bmatrix} A_2 & B_2 \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} C_2 \\ 0 \end{bmatrix}, \quad D_3 = \begin{bmatrix} D_2 \\ 0 \end{bmatrix}.$$

In this system, fundamental RE equilibrium is given by

$$z_{3,t} = \bar{c}_3 + \bar{\Phi}_3 \hat{s}_{t-1} + \bar{\Gamma}_3 u_t, \tag{37}$$

where the coefficient vectors $\bar{c}_3$, $\bar{\Phi}_3$, $\bar{\Gamma}_3$ are determined by

$$\bar{c}_3 = 0_{4\times 1}, \quad A_3\bar{\Phi}_3[0\ 0\ 1\ 0]\bar{\Phi}_3 = \bar{\Phi}_3 - C_3, \quad \bar{\Gamma}_3 = \{I - A_3(\rho I + \bar{\Phi}_3[0\ 0\ 1\ 0])\}^{-1} D_3.$$

Corresponding to this equilibrium, all agents are assumed to be endowed with the PLM of $z_{3,t}$

$$z_{3,t} = c_3 + \Phi_3 \hat{s}_{t-1} + \Gamma_3 u_t. \tag{38}$$

In this PLM the fourth element is

$$\hat{E}_t \hat{\pi}_{t+1} = c_{3,4} + \Phi_{3,4} \hat{s}_{t-1} + \Gamma_{3,4} u_t, \tag{39}$$

where $c_{3,4}$, $\Phi_{3,4}$, and $\Gamma_{3,4}$ are the fourth elements of the vectors c_3, Φ_3, and Γ_3. The forecast $\hat{E}_t z_{3,t+1}$ based on the PLM (38) is given by

$$\hat{E}_t z_{3,t+1} = \begin{bmatrix} \hat{E}_t \hat{\pi}_{t+1} \\ \hat{E}_t \hat{y}_{t+1} \\ \hat{E}_t \hat{s}_{t+1} \\ \hat{E}_t \hat{\pi}_{t+2} \end{bmatrix} = \begin{bmatrix} (c_{3,1} + \Phi_{3,1} c_{3,3}) + \Phi_{3,1}\Phi_{3,3}\hat{s}_{t-1} + (\rho\Gamma_{3,1} + \Phi_{3,1}\Gamma_{3,3})u_t \\ (c_{3,2} + \Phi_{3,2} c_{3,3}) + \Phi_{3,2}\Phi_{3,3}\hat{s}_{t-1} + (\rho\Gamma_{3,2} + \Phi_{3,2}\Gamma_{3,3})u_t \\ (c_{3,3} + \Phi_{3,3} c_{3,3}) + \Phi_{3,3}\Phi_{3,3}\hat{s}_{t-1} + (\rho\Gamma_{3,3} + \Phi_{3,3}\Gamma_{3,3})u_t \\ (c_{3,4} + \Phi_{3,4} c_{3,3}) + \Phi_{3,4}\Phi_{3,3}\hat{s}_{t-1} + (\rho\Gamma_{3,4} + \Phi_{3,4}\Gamma_{3,3})u_t \end{bmatrix}.$$

The first element of this forecast, $\hat{E}_t \hat{\pi}_{t+1}$, is not consistent with (39). Thus, because agents have a PLM for contemporaneous inflation and one for expected inflation next period, they make multiple inflation forecasts in an inconsistent manner. Using the forecast $\hat{E}_t z_{3,t+1}$, the system (36) yields the ALM of $z_{3,t}$

$$z_{3,t} = A_3(I + \Phi_3[0\ 0\ 1\ 0])c_3 + (A_3\Phi_3[0\ 0\ 1\ 0]\Phi_3 + C_3)\hat{s}_{t-1} + \{A_3(\rho I + \Phi_3[0\ 0\ 1\ 0])\Gamma_3 + D_3\}u_t. \tag{40}$$

Thus, the mapping T_3 from the PLM (38) to the ALM (40) can be defined as

$$T_3 \begin{pmatrix} c_3 \\ \Phi_3 \\ \Gamma_3 \end{pmatrix}' = \begin{pmatrix} A_3(I + \Phi_3[0\ 0\ 1\ 0])c_3 \\ A_3\Phi_3[0\ 0\ 1\ 0]\Phi_3 + C_3 \\ A_3(\rho I + \Phi_3[0\ 0\ 1\ 0])\Gamma_3 + D_3 \end{pmatrix}'.$$

Consequently, the E-stability condition for a fundamental RE equilibrium $(\bar{c}_3, \bar{\Phi}_3, \bar{\Gamma}_3)$ is that all eigenvalues of the following three matrices have real parts less than unity.

$$DT_{3,c}(\bar{c}_3, \bar{\Phi}_3, \bar{\Gamma}_3) = A_3(I + \bar{\Phi}_3[0\ 0\ 1\ 0]),$$

$$DT_{3,\Phi}(\bar{c}_3, \bar{\Phi}_3, \bar{\Gamma}_3) = A_3(\bar{\Phi}_{3,3}I + \bar{\Phi}_3[0\ 0\ 1\ 0]),$$

$$DT_{3,\Gamma}(\bar{c}_3, \bar{\Phi}_3, \bar{\Gamma}_3) = A_3(\rho I + \bar{\Phi}_3[0\ 0\ 1\ 0]),$$

where $\bar{\Phi}_{3,3}$ is the third element of the RE equilibrium coefficient vector $\bar{\Phi}_3$.

Under the calibrations of the model parameters presented in Table 1, this E-stability condition generates the same figures as those for representation (I), i.e., Figs. 1 and 3 in the cases of $\sigma_n = 1, 2$. Therefore, in representation (II), if economic agents make inflation forecasts in the aforementioned inconsistent manner, the resulting E-stability region becomes identical with that in representation (I).[13] Because the absence of the consistency in the inflation forecasts is problematic, representation (I) is arguably not appropriate for E-stability analysis.

4 Concluding Remarks

This paper has shown that a pitfall of E-stability analysis can arise in models with multi-period expectations: if an auxiliary variable is introduced as substitute for an expectational endogenous variable in such a model, this shrinks the region of the model parameters that ensure E-stability of a fundamental RE equilibrium. This pitfall has been demonstrated in a simple univariate model with two-period expectations and a DSGE model with trend inflation.[14] Moreover, in the model representation with no auxiliary variable, the same E-stability region as in that with the auxiliary variable has been obtained if economic agents are assumed to make

[13] The equivalence between the E-stability regions is shown formally for the case of an infinite elasticity of labor supply in Appendix B.

[14] The pitfall may be demonstrated in a DSGE model with internal habit formation in consumption preferences, where a two-period-ahead consumption forecast as well as a one-period-ahead one appears in a consumption Euler equation.

multiple forecasts in an inconsistent manner. Therefore, we have argued that the introduction of an auxiliary variable as substitute for an expectational endogenous variable in models with multi-period expectations can induce misleading implications of E-stability that are biased toward E-instability.

Appendix

A Coefficient matrix and vectors of systems (26) and (30)

Let $a_1 = 1/[(1+\sigma_n)(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})\phi_\pi + \alpha\pi^{\theta-1}\{1-\theta\sigma_n(\pi-1)(1-\alpha\beta\pi^{\theta})\}(1+\phi_y)]$. In the system (26), the coefficient matrix A_1 and the coefficient vector C_1 are given by

$$A_1 = \begin{bmatrix} A_{1,11} & A_{1,12} & A_{1,13} & A_{1,14} \\ \frac{1-A_{1,11}\phi_\pi}{1+\phi_y} & \frac{1-A_{1,12}\phi_\pi}{1+\phi_y} & -\frac{A_{1,13}\phi_\pi}{1+\phi_y} & -\frac{A_{1,14}\phi_\pi}{1+\phi_y} \\ \frac{\theta\alpha\pi^{\theta-1}(\pi-1)A_{1,11}}{1-\alpha\pi^{\theta-1}} & \frac{\theta\alpha\pi^{\theta-1}(\pi-1)A_{1,12}}{1-\alpha\pi^{\theta-1}} & \frac{\theta\alpha\pi^{\theta-1}(\pi-1)A_{1,13}}{1-\alpha\pi^{\theta-1}} & \frac{\theta\alpha\pi^{\theta-1}(\pi-1)A_{1,14}}{1-\alpha\pi^{\theta-1}} \\ A_{1,41} & A_{1,42} & A_{1,43} & A_{1,44} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} C_{1,1} \\ -\frac{C_{1,1}\phi_\pi}{1+\phi_y} \\ \alpha\pi^{\theta-1}\left\{\pi + \frac{\theta(\pi-1)C_{1,1}}{1-\alpha\pi^{\theta-1}}\right\} \\ 0 \end{bmatrix},$$

where

$$A_{1,11} = a_1\{\alpha\pi^{\theta-1}(\beta + A_{1,41})(1+\phi_y) + (1+\sigma_n)(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})\},$$

$$A_{1,12} = a_1\{\alpha\pi^{\theta-1}A_{1,42}(1+\phi_y) + (1+\sigma_n)(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})\},$$

$$A_{1,13} = a_1\alpha\pi^{\theta-1}A_{1,43}(1+\phi_y), \quad A_{1,14} = a_1\alpha\pi^{\theta-1}A_{1,44}(1+\phi_y),$$

$$A_{1,41} = \theta\beta(\pi-1)(1-\alpha\pi^{\theta-1}), \quad A_{1,42} = \beta(\pi-1)(1+\sigma_n)(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta}),$$

$$A_{1,43} = \sigma_n\beta(\pi-1)(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta}), \quad A_{1,44} = \alpha\beta\pi^{\theta},$$

$$C_{1,1} = a_1\sigma_n\alpha\pi^{\theta}(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})(1+\phi_y).$$

In the system (30), the coefficient matrix A_2 and the coefficient vectors B_2, C_2 are given by

$$A_2 = \begin{bmatrix} A_{2,11} & A_{2,12} & A_{2,13} \\ \frac{1-A_{2,11}\phi_\pi}{1+\phi_Y} & \frac{1-A_{2,12}\phi_\pi}{1+\phi_y} & -\frac{A_{2,13}\phi_\pi}{1+\phi_y} \\ \frac{\theta\alpha\pi^{\theta-1}(\pi-1)A_{2,11}}{1-\alpha\pi^{\theta-1}} & \frac{\theta\alpha\pi^{\theta-1}(\pi-1)A_{2,12}}{1-\alpha\pi^{\theta-1}} & \frac{\theta\alpha\pi^{\theta-1}(\pi-1)A_{2,13}}{1-\alpha\pi^{\theta-1}} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} B_{2,1} \\ -\frac{B_{2,1}\phi_\pi}{1+\phi_y} \\ \frac{\theta\alpha\pi^{\theta-1}(\pi-1)B_{2,1}}{1-\alpha\pi^{\theta-1}} \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{2,1} \\ -\frac{C_{2,1}\phi_\pi}{1+\phi_y} \\ \alpha\pi^{\theta-1}\left\{\pi + \frac{\theta(\pi-1)C_{2,1}}{1-\alpha\pi^{\theta-1}}\right\} \end{bmatrix},$$

where

$$A_{2,11} = a_1[\alpha\beta\pi^{\theta-1}\{1+\alpha\pi^{\theta}+\theta(\pi-1)(1-\alpha\pi^{\theta-1})\}(1+\phi_y)+(1+\sigma_n)(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})],$$
$$A_{2,12} = a_1(1+\sigma_n)(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})\{1-\alpha\beta\pi^{\theta-1}(1+\phi_y)\},$$
$$A_{2,13} = -a_1\sigma_n\alpha\beta\pi^{\theta-1}(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})(1+\phi_y),$$
$$B_{2,1} = -a_1\alpha\beta\pi^{\theta-1}\alpha\beta\pi^{\theta}(1+\phi_y), \quad C_{2,1} = a_1\sigma_n\alpha\pi^{\theta}(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta})(1+\phi_y).$$

B The case of an infinite elasticity of labor supply

In the case of an infinite elasticity of labor supply (i.e., $\sigma_n = 0$), the E-stability conditions for the representations (I) and (II) presented in Section 3.5 generate Figs. 5 and 6 under the calibrations of the other model parameters presented in Table 1. In line with the cases of $\sigma_n = 1, 2$, these figures show that E-stability of a fundamental RE equilibrium is more likely in representation (II) than in representation (I). Note that in the case of $\sigma_n = 0$, the Taylor principle (35) is a necessary condition for E-stability as well as determinacy even in representation (II).

In representation (II), if economic agents make inflation forecasts in the inconsistent manner mentioned in Section 3.8, the resulting E-stability region becomes identical with that in representation (I). This is shown in the following proposition.

Proposition 1 *Suppose that $\sigma_n = 0$. Then the same E-stability region as in representation (I) is obtained in representation (II) if agents' inflation forecasting is inconsistent in the aforementioned sense.*

Proof *The exposition in Section 3.5 implies that for representation (I), the fundamental RE equilibrium is E-stable if and only if all eigenvalues of the matrices A_1^0 and ρA_1^0 have real parts less than unity, where*

$$A_1^0 = \begin{bmatrix} A_{1,11} & A_{1,12} & A_{1,14} \\ \frac{1-A_{1,11}\phi_\pi}{1+\phi_y} & \frac{1-A_{1,12}\phi_\pi}{1+\phi_y} & -\frac{A_{1,14}\phi_\pi}{1+\phi_y} \\ A_{1,41} & A_{1,42} & A_{1,44} \end{bmatrix}$$

is the coefficient matrix A_1 in the case of $\sigma_n = 0$. Likewise, for representation (II) in which agents' inflation forecasting is inconsistent in the aforementioned sense, the fundamental RE equilibrium is E-stable if and only if all eigenvalues of the matrices A_3^0 and ρA_3^0 have real parts *less than unity, where*

$$A_3^0 = \begin{bmatrix} A_{2,11} & A_{2,12} & B_{2,1} \\ \frac{1-A_{2,11}\phi_\pi}{1+\phi_y} & \frac{1-A_{2,12}\phi_\pi}{1+\phi_y} & -\frac{B_{2,1}\phi_\pi}{1+\phi_y} \\ 1 & 0 & 0 \end{bmatrix}$$

is the coefficient matrix A_3 *in the case of* $\sigma_n = 0$*. It is straightforward to verify that the characteristic equations of* A_1^0 *and* A_3^0 *are identical and given by*

$$r^3 - \left(A_{2,11} + \frac{1 - A_{2,12}\phi_\pi}{1+\phi_y}\right) r^2 + \left(\frac{A_{2,11} - A_{2,12}}{1+\phi_y} - B_{1,2}\right) r + \frac{B_{1,2}}{1+\phi_y} = 0.$$

Therefore, the eigenvalues of A_1^0 *and* A_3^0 *must be identical and the eigenvalues of* ρA_1^0 *and* ρA_3^0 *must be identical.* ■

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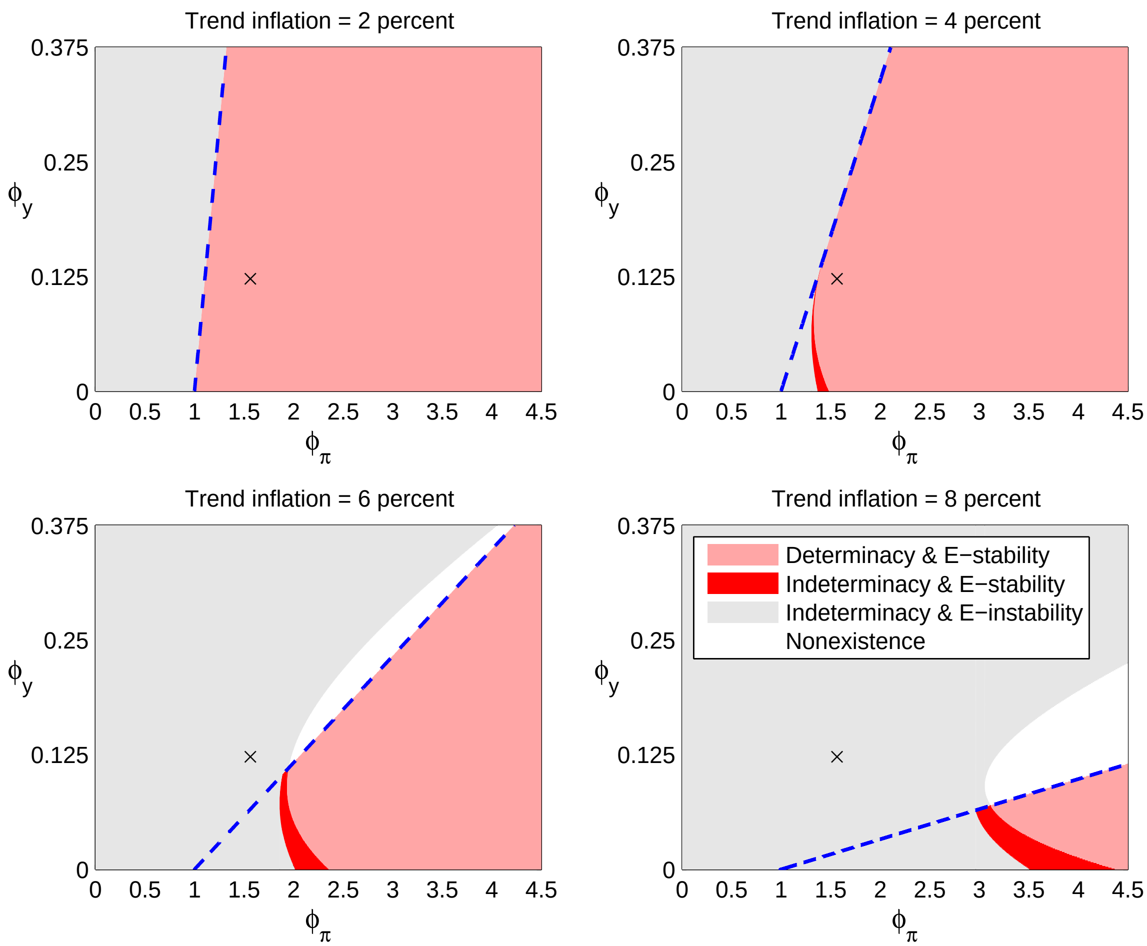
Table 1: Calibrations of parameters for the quarterly model

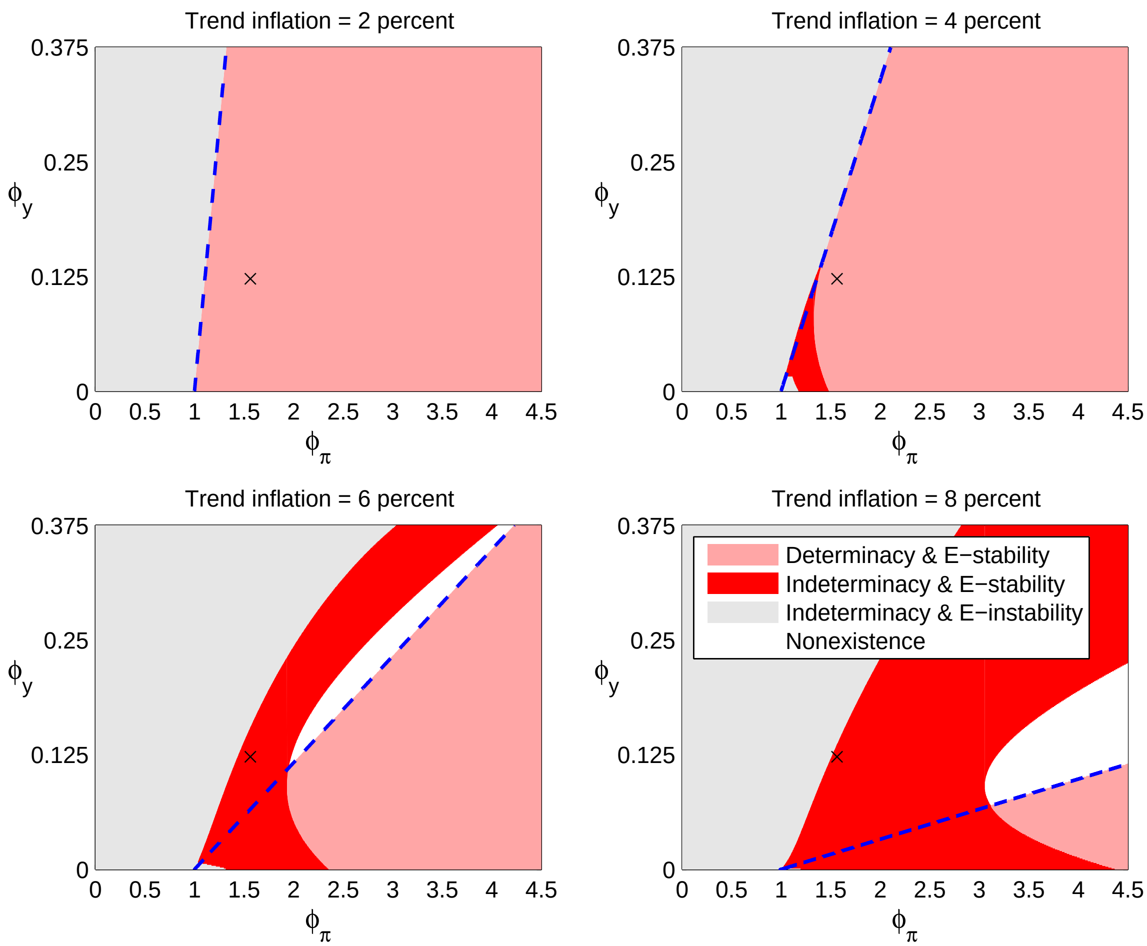
β	subjective discount factor	0.99
α	probability of no price adjustment	0.75
θ	price elasticity of demand for differentiated goods	11
π	gross trend inflation rate	1.005, 1.010, 1.015, 1.020
σ_n	inverse of elasticity of labor supply	1, 2
ρ	shock persistence	0.8

Figure 1: Regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium: Representation (I) with the elasticity of labor supply of unity (i.e., $\sigma_n = 1$).



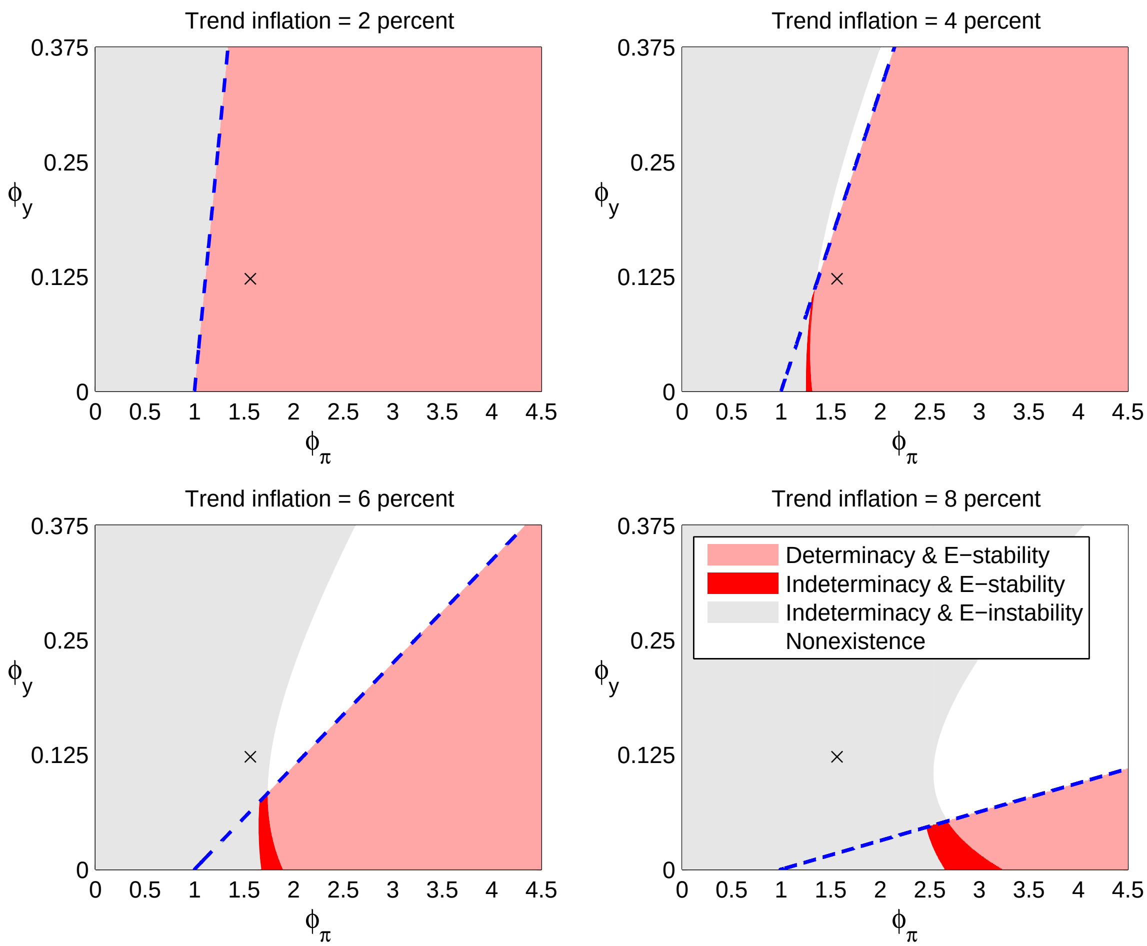
Note: In each panel the mark "×" shows the Taylor (1993) estimates $(\phi_\pi, \phi_y) = (1.5, 0.125)$ and the dashed line represents the boundary given by the long-run version of the Taylor principle (35).

Figure 2: Regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium: Representation (II) with the elasticity of labor supply of unity (i.e., $\sigma_n = 1$).



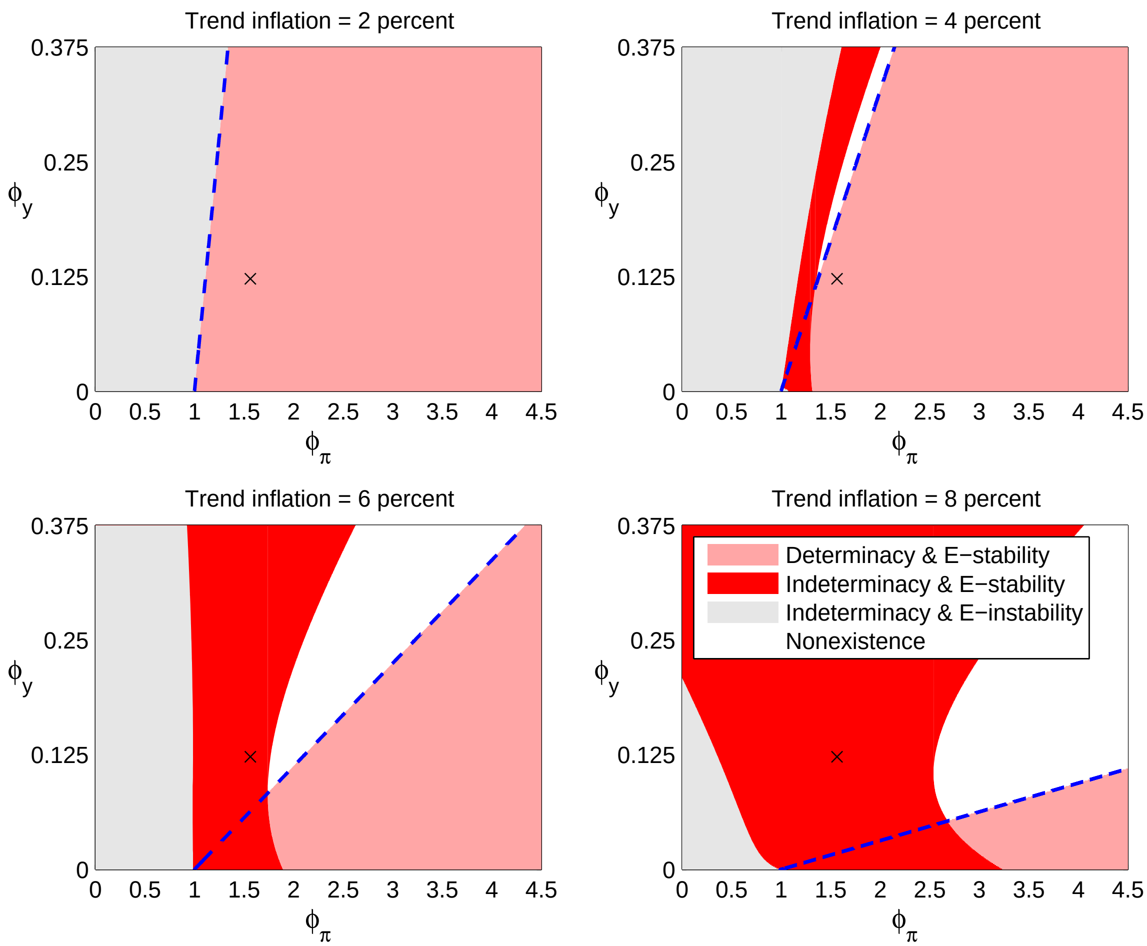
Note: In each panel the mark "×" shows the Taylor (1993) estimates $(\phi_\pi, \phi_y) = (1.5, 0.125)$ and the dashed line represents the boundary given by the long-run version of the Taylor principle (35).

Figure 3: Regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium: Representation (I) with the elasticity of labor supply of one half (i.e., $\sigma_n = 2$).



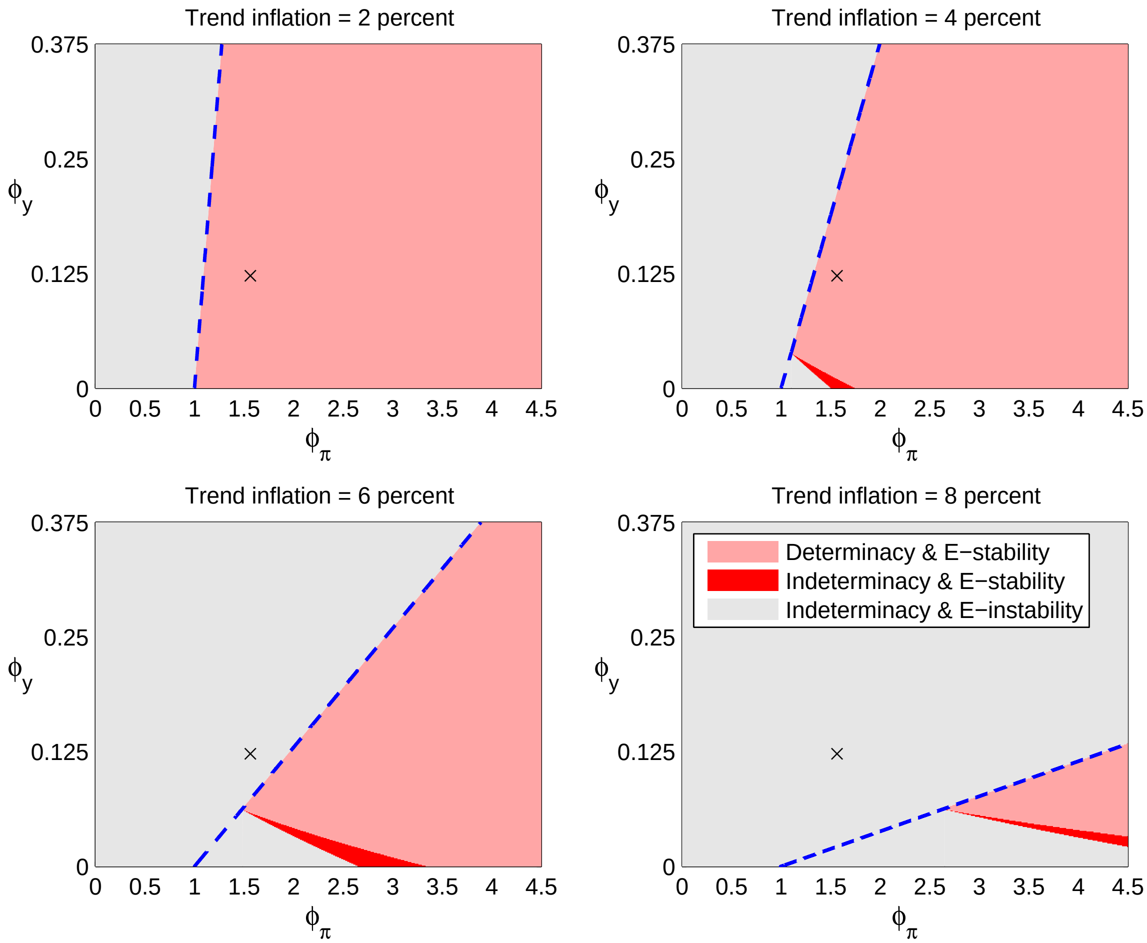
Note: In each panel the mark "×" shows the Taylor (1993) estimates $(\phi_\pi, \phi_y) = (1.5, 0.125)$ and the dashed line represents the boundary given by the long-run version of the Taylor principle (35).

Figure 4: Regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium: Representation (II) with the elasticity of labor supply of one half (i.e., $\sigma_n = 2$).



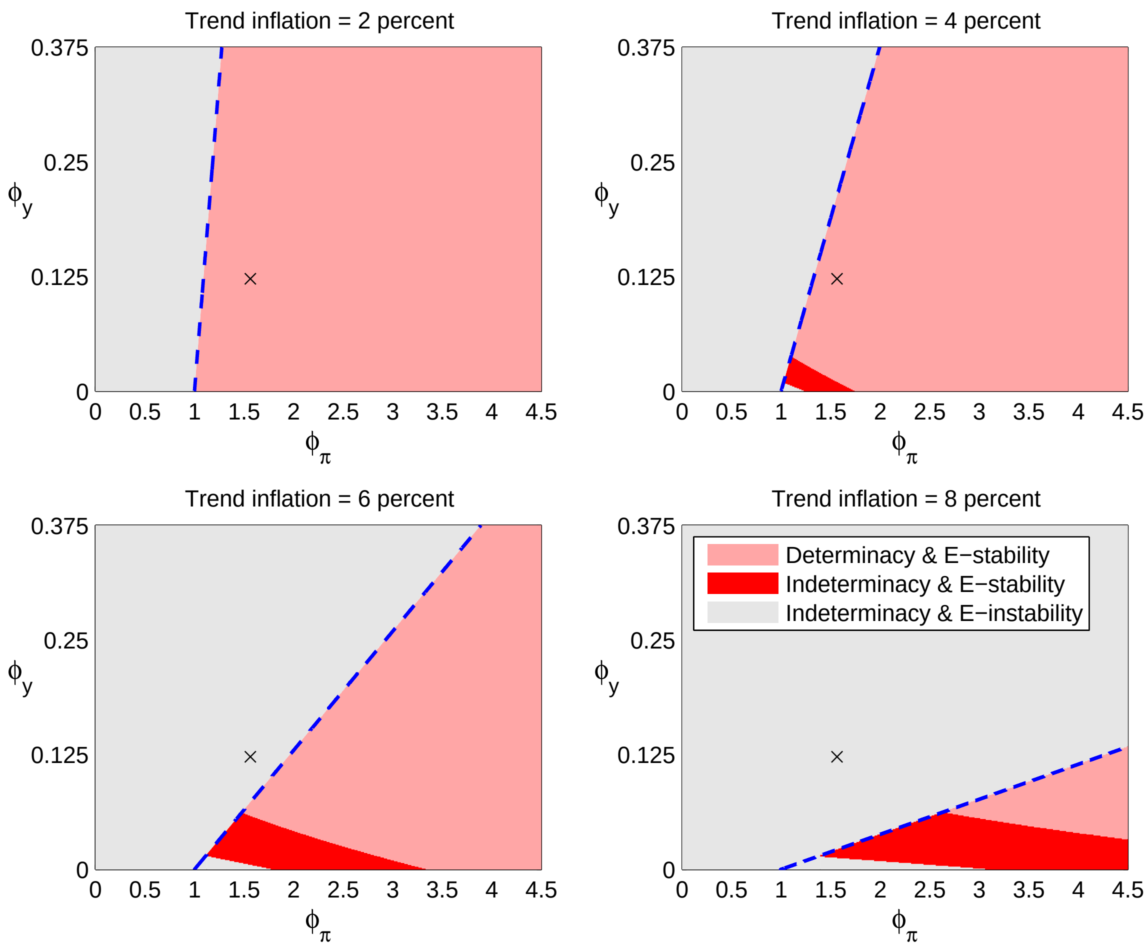
Note: In each panel the mark "×" shows the Taylor (1993) estimates $(\phi_\pi, \phi_y) = (1.5, 0.125)$ and the dashed line represents the boundary given by the long-run version of the Taylor principle (35).

Figure 5: Regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium: Representation (I) with an infinite elasticity of labor supply (i.e., $\sigma_n = 0$).



Note: In each panel the mark "×" shows the Taylor (1993) estimates $(\phi_\pi, \phi_y) = (1.5, 0.125)$ and the dashed line represents the boundary given by the long-run version of the Taylor principle (35).

Figure 6: Regions of the Taylor rule's coefficients (ϕ_π, ϕ_y) that ensure E-stability of a fundamental RE equilibrium as well as determinacy of RE equilibrium: Representation (II) with an infinite elasticity of labor supply (i.e., $\sigma_n = 0$).



Note: In each panel the mark "×" shows the Taylor (1993) estimates $(\phi_\pi, \phi_y) = (1.5, 0.125)$ and the dashed line represents the boundary given by the long-run version of the Taylor principle (35).