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Abstract

Under linear approximations for asset prices and the assumption of independence between expected consumption growth and time-varying volatility, long-run risks models imply constant market prices of risks and often generate counterfactual results about asset return and cash flow predictability. We develop and estimate a nonlinear equilibrium asset pricing model with recursive preferences and a flexible econometric specification of cash flow processes. While in many long-run risks models time-varying volatility influences only risk premium but not expected cash flows, in our model a common set of risk factors drive both expected cash flow and risk premium dynamics. This feature helps the model to overcome two main criticisms against long-run risk models following Bansal and Yaron (2004): the over-predictability of cash flows by asset prices and the tight relation between time-varying risk premia and growth volatility. Our model extends the approach in Le and Singleton (2010) to a setting with multiple cash flows. We estimate the model using the long-run historical data in the U.S. and find that the model with generalized market prices of risks produces cash flow and return predictability that are more consistent with the data.

Key Words: Long-run consumption risks; Time-varying risk premium; Recursive preferences

JEL Classification: G12, E21

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1 Introduction

Equilibrium models of asset pricing relate current-period values of claims to future cash flows to their exposures to fundamental macroeconomic risks borne by investors. In consumption-based models featuring an Epstein-Zin-Weil recursive utility function (e.g. Kandel and Stambaugh 1991, Bansal and Yaron 2004, Restoy and Weil 2011 among others), uncertainty about long-run macroeconomic growth is emphasized as a primary risk factor. In these long-run risks models, changes in expected consumption and dividend growth play important roles in asset valuation. Since asset prices are highly persistent while observed consumption growth and dividend growth are not, identifying consumption risk factors that can explain movements in asset prices and cash flow data at the same time is challenging. Explaining asset prices demands highly persistent cash flow risks while matching observed cash flow data requires that their persistence is low.

Standard long-run risk models resolve this issue by assuming that observed consumption growth is the sum of a small but highly persistent component and a large but serially uncorrelated component. The resulting small signal-to-noise ratio from observed cash flow data to the predictable component, however, makes empirical identification difficult, and teasing out such “easy-to-care” but “hard-to-measure” risks are quite sensitive to econometric specifications.¹ Often these econometric specifications lead to implications of return and cash flow predictability that are inconsistent with empirical evidence. For instance, as pointed out by Beeler and Campbell (2012), the predictability of consumption and dividend growth from asset prices implied by the long-run risk model in Bansal and Yaron (2004) is much higher than in the data. In addition, these models also imply that priced risk factors that predict future cash flow are distinct from those predicting excess stock market returns. Lettau and Ludvigson (2005), however, documents strong evidence of a common component in time-varying expected dividend growth and risk premium of the aggregate stock market.

One reason that long-run risk models lead to counterfactual implications for predictability is the assumption of constant market prices of risks. With constant market prices of risks, long-run risks models attribute time-variations in risk premia entirely to changes in the quantity of risk (or consumption and dividend growth volatility).

¹Schorfheide et al. (2014) argue that estimation with only cash flow data also support the existence of small but predictable components in cash flows. However, they impose a fairly tight prior for the signal-to-noise ratio and the posterior of that parameter is close to the upper end of the prior distribution. It remains to be seen if their result will be robust to a relatively uninformative prior for the signal-to-noise ratio parameter.

Predictable consumption growth, while a major risk factor driving asset prices, has no impact on the *dynamics* of risk premia.

In this paper we try to resolve this challenge to long-run risks models by using a more flexible econometric specification of market prices of risks and expected cash flows. We show that constant market price of risk is not an inherent feature of long-run risks models, but rather the artificial result of restricting asset price solutions as linear functions of state variables. We allow small and predictable components in consumption growth and dividend growth with stochastic volatility and assume recursive preferences for the representative agent as common in long-run risks models. However, we use a quadratic approximation for the wealth-consumption ratio instead of the standard linear approximation. In the context of recursive preferences, this change makes market prices of risks time-varying. Time-varying market prices of risks introduce additional sources of time-variation in risk premia and help to avoid the restrictive dichotomy between expected cash flow and risk premium dynamics found in most long-run risk models. In our model expected cash flows and risk premia are driven by a common set of risk factors. Our model also relaxes the restrictive assumption of independence between expected cash flows and time-varying volatility so that macroeconomic uncertainty can have a first-order effect on consumption and dividend growth.²

Our modeling strategy follows closely that of Le and Singleton (2010). Different from the standard approach in long-run risks models, we treat market prices of risks as free parameters and reverse engineer the cash flow dynamics under equilibrium asset pricing restrictions. Standard long-run risks models start from tight restrictions on the evolution of expected cash flows under the real-world probability measure and derive market prices of risks as functions of parameters governing investors' preferences and cash flow processes using equilibrium no-arbitrage conditions. In contrast we take an agnostic approach to the econometric model of cash flow processes. In our setting, functional forms of cash flow processes under the real-world probability measure are not specified ex-ante but are derived from equilibrium no-arbitrage conditions. The main advantage of this reverse-engineering approach is to allow greater econometric flexibility of the asset pricing model that follows. For example, the standard long-run risks models turn out to be special cases of our model when we impose additional restrictions on market prices of risks and the parameters governing cash flow processes

²Bloom (2009) shows that time-varying uncertainty can have a first-order impact on the aggregate output in a general equilibrium macro model. Bansal et al. (2014) show that time-varying volatility can predict consumption growth and asset prices together using a vector-autoregression model.

under the risk-neutral probability measure.³ In essence, we let market prices of risks to be free parameters in order to have a more flexible econometric specification for cash flow processes while retaining a tractable closed-form solution for asset prices.⁴

Empirical analysis of the U.S. data vindicates that the flexible specification brings the long-run risk model further closer to the data with regard to cash flow and asset return predictability as well as risk premium dynamics. With time-varying market prices of risks, the model can match the fact that the price/dividend ratio mainly predicts excess market return than cash flows especially at long horizons. Also, by introducing an additional channel for time-varying risk premium that is not proportional to time-varying volatility of cash flows, the model can generate more plausible volatility of equity risk premium. At the same time, the model estimates are consistent with cyclical patterns of cash flows and asset returns documented in other literature.

Our paper contributes to the empirical study of consumption-based asset pricing models with recursive preferences and predictable cash flows.⁵ Following the seminal work of Bansal and Yaron (2004), a growing body of literature has examined empirically the role of a small but persistent component in consumption and dividend growth at explaining stock market returns when the representative agent is characterized by an Epstein-Zin-Weil recursive utility function. These studies include Bansal, Dittmar and Lundbald (2005), Bansal, Gallant and Tauchen (2007), Hansen, Heaton and Li (2008), Bansal, Dittmar and Kiku (2009), Constantinides and Ghosh (2011), Bansal, Kiku and Yaron (2012), Beeler and Campbell (2012), Yu (2012), Schorfheide, Song and Yaron (2014) and Belo et al. (2015) among others. One challenge to the quantitative analysis of these long-run risk models is that the wealth-consumption ratio, a key variable in the utility function, is unobservable. The standard approach in the existing literature is to approximate the wealth-consumption ratio as a linear function of the underlying state variables. This linear approximation often combined with restrictive econometric specifications of the cash flow processes (e.g. constant leverage ratio) lead to closed-form solutions to asset prices which make empirical analysis tractable. Our paper extends the approach in Le and Singleton (2010) to a setting of multiple cash flows. This extension

³Our model can also replicate the main features of habit-formation model of Campbell and Cochrane (1999) in which consumption growth is i.i.d. but market price of risk is time-varying.

⁴Even though we treat the market prices of risk as free parameters in the model, they still have clear economic interpretations. They measure the sensitivity of investor's consumption claim to fundamental shocks. In addition, our specification distinguishes volatility shocks from non-volatility shocks.

⁵See Campbell (2003) and Cochrane (2007) for excellent reviews of consumption-based asset pricing models. Ludvigson (2012) provide a survey of recent empirical studies of consumption-based asset pricing models.

allows us to retain the tractability of the standard long-run risk models with much more flexible econometric specifications of the cash flow processes such as time-varying leverage ratios.⁶

More importantly, the wealth-consumption ratio in our model can be a nonlinear function of the state variables. This nonlinearity implies time-varying market prices of risks. As a result, risk premium is time-varying even if the quantity of risk (i.e. volatility of consumption growth) is constant. In contrast, most long-run risks models rely exclusively on time-varying quantity of risk as the source of time-varying risk premia.⁷ We show that a time-varying market price of risk is critical at reconciling long-run risk models with empirical facts about cash flow *and* return predictability.

Our paper is organized as follows. In the next section we describe the equilibrium asset pricing model used in the subsequent empirical analysis. Section 3 explains the data used in this paper and the econometric methodology. Section 4 discusses the main empirical results from the estimated equilibrium asset pricing models and Section 5 concludes.

2 A Long-run Risks Model with Time-varying Market Price of Risk

Below we explain the main assumptions and the setup of the long-run risks model with time-varying market prices of risks. To facilitate empirical analysis, we subsequently derive the joint likelihood function of cash flow and asset return data. Detailed derivations and specific cases of the general version of this model can be found in the Appendix.

⁶Belo et al. (2015) shows that a stationary leverage ratio is key at reproducing the downward-sloping term structure of equity risk premia in the long-run risk model. To make the price-dividend ratio an exponential affine function of state variables, they restrict the functional form of the log leverage ratio to be linear with respect to state variables. While more flexible than the original Bansal and Yaron (2004), they still impose the restrictive assumption on the leverage process to obtain the affine log price-dividend ratio.

⁷One exception is Creal and Wu (2015) who generate time-varying bond risk premium by introducing a preference shock that has time-varying sensitivity with respect to innovations to long-run consumption and volatility risks. Much like Campbell and Cochrane (1999), this time-varying sensitivity function generates the time-varying market price of risk in the stochastic discount factor. In general, this would imply nonlinear functions for log price/cash flow ratios. Creal and Wu (2015) restrict the drift term of a preference shock to make equilibrium asset pricing restrictions compatible with the linear approximation of log price/cash flow ratios. However, they do not empirically evaluate the validity of this restriction on the drift term of the preference shock.

2.1 Dynamics of State Variables

We assume that all the state variables relevant for asset pricing are summarized by an $n \times 1$ Markovian vector, X_t . Under this assumption, equilibrium asset pricing models would determine asset prices as functions of X_t . For assets with trend growth in cash flows, price/cash flow ratios would be given as functions of X_t . Without loss of generality, we consider a class of asset pricing models where a log price/cash flow ratio, z_t , is described by a function of X_t ,

$$z_t = f(X_t). \quad (1)$$

While $f(X_t)$ can be nonlinear, we will use the following approximation for the nonlinear asset pricing function following Le and Singleton (2010) to simplify the analysis:

$$\Delta z_{t+1} \approx \Gamma(X_t)' \Delta X_{t+1}, \quad (2)$$

where $\Gamma(X_t) = \frac{\partial f(X_t)}{\partial X_t}$ and $\Delta X_{t+1} = X_{t+1} - X_t$.

Returns of a claim to (log) cash flow, y_t , can be obtained by the Campbell-Shiller (1988) log-linear approximation:⁸

$$r_{t+1} = k_0 - k_1 z_t + k_2 \Delta z_{t+1} + \Delta y_{t+1}. \quad (3)$$

where $k_1 = 1 - k_2$, $k_2 = \frac{e^{\bar{z}}}{1+e^{\bar{z}}}$ and $k_0 = \ln(1 + e^{\bar{z}}) - \frac{\bar{z}e^{\bar{z}}}{1+e^{\bar{z}}}$. \bar{z} is the steady-state value of z_t and it is given by $f(\bar{X})$.

We consider two cash flows, log consumption, c_t , and log dividend of S&P 500 stock market index, d_t . The normalized prices of the claims to these two cash flows are, respectively,

$$z_{c,t} = f_c(X_t), \quad (4)$$

$$z_{d,t} = f_d(X_t), \quad (5)$$

with holding-period returns given by:

$$r_{c,t+1} = k_{c,0} - k_{c,1} z_{c,t} + k_{c,2} \Delta z_{c,t+1} + \Delta c_{t+1}, \quad (6)$$

$$r_{d,t+1} = k_{d,0} - k_{d,1} z_{d,t} + k_{d,2} \Delta z_{d,t+1} + \Delta d_{t+1}, \quad (7)$$

⁸In the data, the approximation error from the linear formula is extremely small. Moreover, in our estimation we will use returns constructed according to the Campbell-Shiller linear present value formula.

or approximately,

$$r_{c,t+1} = k_{c,0} - k_{c,1}f_c(X_t) + k_{c,2}\Gamma_c(X_t)'\Delta X_{t+1} + \Delta c_{t+1}, \quad (8)$$

$$r_{d,t+1} = k_{d,0} - k_{d,1}f_d(X_t) + k_{d,2}\Gamma_d(X_t)'\Delta X_{t+1} + \Delta d_{t+1}. \quad (9)$$

2.2 Investor Preferences

We assume investors are endowed with Epstein-Zin (1989) recursive preferences in which the log stochastic discount factor (m_{t+1}) is given by

$$-m_{t+1} = -\theta \log \delta + \frac{\theta}{\psi} \Delta c_{t+1} - (\theta - 1)r_{c,t+1}, \quad (10)$$

we then have

$$\begin{aligned} -m_{t+1} &= -\theta \log \delta - (\theta - 1)k_{c,0} + (\theta - 1)k_{c,1}f_c(X_t) \\ &\quad - (\theta - 1)k_{c,2}\Gamma_c(X_t)'\Delta X_{t+1} + \gamma \Delta c_{t+1}. \end{aligned} \quad (11)$$

In the utility function, $0 < \delta < 1$ is the time discount factor, $\gamma > 0$ is the parameter of risk-aversion, $\psi > 0$ is the parameter of intertemporal elasticity of substitution and $\theta = \frac{1-\gamma}{1-1/\psi}$. $r_{c,t+1}$ is the return on the asset that pays aggregate consumption each period as its dividend.

2.3 Cash Flow Dynamics under Asset Pricing Restrictions

The utility function imposes equilibrium restrictions on asset returns. These restrictions can be conveniently expressed as moment conditions under the risk neutral probability. In particular, we can define the risk-neutral probability measure, \mathbb{Q} , by the following Radon-Nikodym derivative,

$$\xi_{t,t+1} = \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)_{t,t+1} = \frac{e^{m_{t+1}}}{E_t^P(e^{m_{t+1}})}. \quad (12)$$

where E_t^P denotes conditional expectation under the physical probability measure \mathbb{P} .

Let $r_{f,t}$ be the risk-free interest rate, then

$$e^{r_{f,t}} = \frac{1}{E_t^P(e^{m_{t+1}})} = E_t^P(\xi_{t,t+1}e^{-m_{t+1}}) = E_t^Q(e^{-m_{t+1}}), \quad (13)$$

and in equilibrium we must have

$$e^{r_{f,t}} = E_t^{\mathbb{Q}}(e^{r_{c,t+1}}) = E_t^{\mathbb{Q}}(e^{r_{d,t+1}}), \quad (14)$$

where $E_t^{\mathbb{Q}}$ denotes conditional expectation under the risk-neutral probability measure \mathbb{Q} defined in (12).

We suppose that, under the risk-neutral probability measure \mathbb{Q} ,

$$\Delta c_{t+1} = \tilde{g}_c(X_t) + \sigma_c(X_t) \tilde{\varepsilon}_{c,t+1}, \quad (15)$$

$$\Delta d_{t+1} = \tilde{g}_d(X_t) + \sigma_d(X_t) \tilde{\varepsilon}_{d,t+1}, \quad (16)$$

$$\Delta X_{t+1} = \tilde{\Phi}(X_t) + \Sigma(X_t) \tilde{\eta}_{t+1}, \quad (17)$$

where $\tilde{\varepsilon}_{c,t+1}$, $\tilde{\varepsilon}_{d,t+1}$ and $\tilde{\eta}_{t+1}$ are all multivariate standard normal under \mathbb{Q} .⁹ Let ρ_{cd} be the correlation coefficient between $\tilde{\varepsilon}_{c,t+1}$ and $\tilde{\varepsilon}_{d,t+1}$. We assume that $\tilde{\varepsilon}_{c,t+1}$, $\tilde{\varepsilon}_{d,t+1}$ are independent of $\tilde{\eta}_{t+1}$.¹⁰ Moreover, let $\Omega(X_t) = \Sigma(X_t)' \Sigma(X_t)$. Most long-run risks models typically assume that $\tilde{g}_c(X_t)$ and $\tilde{g}_d(X_t)$ are linear functions of X_t .¹¹ Then they use asset pricing restrictions to determine coefficients in log price/cash flow ratios. In contrast, we assume specific functional forms of log price/cash flow ratios and back out expectations of cash flows using asset pricing restrictions. Hence, if we use quadratic functions to approximate log price/cash flow ratios, $\tilde{g}_c(X_t)$ and $\tilde{g}_d(X_t)$ that satisfy asset pricing restrictions can be nonlinear functions of X_t .

Under the assumption of the multivariate normality of $\tilde{\varepsilon}_{c,t+1}$, $\tilde{\varepsilon}_{d,t+1}$ and $\tilde{\eta}_{t+1}$ conditional on X_t , the equilibrium conditions, (13) and (14), impose cross-equation restrictions on the conditional mean of consumption and dividend growth (under the risk neutral probability measure).

$$\begin{aligned} E_t^{\mathbb{Q}}(e^{-m_{t+1}}) &= e^{-\theta \ln \delta - (\theta-1)k_{c,0} + (\theta-1)k_{c,1}f_c(X_t)} \\ &\times e^{-(\theta-1)k_{c,2}\Gamma_c(X_t)' \tilde{\Phi}(X_t) + \gamma \tilde{g}_c(X_t)} \\ &\times e^{\frac{1}{2}(\theta-1)^2 k_{c,2}^2 \Gamma_c(X_t)' \Omega(X_t) \Gamma_c(X_t) + \frac{1}{2} \gamma^2 \sigma_c^2(X_t)}, \end{aligned} \quad (18)$$

⁹When volatility risk factors are included, it might be necessary to have gamma distributions for some part of η_{t+1} in order to guarantee the positivity of volatility. However, many long-run risks models originated from Bansal and Yaron (2004) assume normal distribution for stochastic volatility because the approximation error of truncating negative variances is typically small. We adopt this approach too.

¹⁰It is a straight forward extension to allow $\tilde{\varepsilon}_{c,t+1}$ and $\tilde{\varepsilon}_{d,t+1}$ to be correlated with $\tilde{\eta}_{t+1}$.

¹¹One exception is Belo et. al. (2015) who allow dividend growth to be nonlinear functions of expected consumption growth through a time-varying leverage ratio.

$$\begin{aligned}
E_t^Q(e^{r_{c,t+1}}) &= e^{k_{c,0} - k_{c,1}f_c(X_t)} \\
&\times e^{k_{c,2}\Gamma_c(X_t)' \tilde{\Phi}(X_t) + \tilde{g}_c(X_t)} \\
&\times e^{\frac{1}{2}k_{c,2}^2\Gamma_c(X_t)'\Omega(X_t)\Gamma_c(X_t) + \frac{1}{2}\sigma_c^2(X_t)},
\end{aligned} \tag{19}$$

$$\begin{aligned}
E_t^Q(e^{r_{d,t+1}}) &= e^{k_{d,0} - k_{d,1}f_d(X_t)} \\
&\times e^{k_{d,2}\Gamma_d(X_t)' \tilde{\Phi}(X_t) + \tilde{g}_d(X_t)} \\
&\times e^{\frac{1}{2}k_{d,2}^2\Gamma_d(X_t)'\Omega(X_t)\Gamma_d(X_t) + \frac{1}{2}\sigma_d^2(X_t)}.
\end{aligned} \tag{20}$$

In equilibrium, $E_t^Q(e^{-m_{t+1}}) = E_t^Q(e^{r_{c,t+1}}) = E_t^Q(e^{r_{d,t+1}})$, it then follows that

$$\begin{aligned}
\tilde{g}_c(X_t) &= -\frac{\theta}{1-\gamma}[\ln \delta + k_{c,0} - k_{c,1}f_c(X_t) + k_{c,2}\Gamma_c(X_t)' \tilde{\Phi}(X_t)] \\
&- \frac{\theta(2-\theta)}{2(1-\gamma)}k_{c,2}^2\Gamma_c(X_t)'\Omega(X_t)\Gamma_c(X_t) - \frac{1+\gamma}{2}\sigma_c^2(X_t),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\tilde{g}_d(X_t) - \tilde{g}_c(X_t) &= (k_{c,0} - k_{d,0}) - [k_{c,1}f_c(X_t) - k_{d,1}f_d(X_t)] \\
&+ [k_{c,2}\Gamma_c(X_t) - k_{d,2}\Gamma_d(X_t)]' \tilde{\Phi}(X_t) + \frac{1}{2}[\sigma_c^2(X_t) - \sigma_d^2(X_t)] \\
&+ \frac{1}{2}[k_{c,2}^2\Gamma_c(X_t)'\Omega(X_t)\Gamma_c(X_t) - k_{d,2}^2\Gamma_d(X_t)'\Omega(X_t)\Gamma_d(X_t)].
\end{aligned} \tag{22}$$

Notice that second moments of state variables as well as market prices of risks implied by these state variables directly influence conditional means of consumption growth and dividend growth under the risk-neutral measure. We have not taken any stance on the precise interpretation of X_t yet. Since there are two asset pricing restrictions, we can pin down $\tilde{g}_c(X_t)$ and $\tilde{g}_d(X_t)$ by assuming functional forms for all the other variables showing up in the cash flow dynamics. For example, we can assume $\sigma_c(X_t)$, $\sigma_d(X_t)$, and $\Omega(X_t)$ are linear functions of some variables in X_t . In addition, we can assume $\tilde{\Phi}(X_t)$, $f_c(X_t)$, and $f_d(X_t)$ are linear with respect to X_t . In this case, we can determine conditional means of consumption growth and dividend growth as affine functions of X_t .

To estimate our model, we need to derive the likelihood function of asset prices and cash flows that respects these equilibrium conditions under the physical probability measure. Let $f^P(\cdot|X_t)$ and $f^Q(\cdot|X_t)$ denote the conditional probability density function (pdf) of a random variable under physical and risk-neutral probability measures

respectively. It then follows that

$$f^P(\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1}|X_t) = \frac{e^{-m_{t+1}}}{E_t^Q(e^{-m_{t+1}})} f^Q(\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1}|X_t), \quad (23)$$

where

$$\begin{aligned} \frac{e^{-m_{t+1}}}{E_t^Q(e^{-m_{t+1}})} &= e^{-\frac{1}{2}(\theta-1)^2 k_{c,2}^2 \Gamma_c(X_t)' \Omega(X_t) \Gamma_c(X_t) - \frac{1}{2} \gamma^2 \sigma_c^2(X_t)} \\ &\times e^{-(\theta-1) k_{c,2} \Gamma_c(X_t)' \Sigma(X_t) \tilde{\eta}_{t+1} + \gamma \sigma_c(X_t) \tilde{\varepsilon}_{c,t+1}}. \end{aligned} \quad (24)$$

Using the above probability measure transformation, we derive the dynamics of cash flows and state variables under the physical probability measure \mathbb{P} ,

$$\Delta c_{t+1} = \tilde{g}_c(X_t) + \gamma \sigma_c(X_t)^2 + \sigma_c(X_t) \varepsilon_{c,t+1}, \quad (25)$$

$$\Delta d_{t+1} = \tilde{g}_d(X_t) + \gamma \rho_{cd} \sigma_c(X_t) \sigma_d(X_t) + \sigma_d(X_t) \varepsilon_{d,t+1}, \quad (26)$$

$$\Delta X_{t+1} = \tilde{\Phi}(X_t) - (\theta - 1) \kappa_{c,2} \Omega(X_t) \Gamma_c(X_t) + \Sigma(X_t) \eta_{t+1} = \Phi(X_t) + \Sigma(X_t) \eta_{t+1}, \quad (27)$$

where $\varepsilon_{c,t+1}$, $\varepsilon_{d,t+1}$, and η_{t+1} are standard normal random variables under \mathbb{P} . Notice that the conditional mean of consumption increases proportional to risk aversion and volatility of innovations to unexpected consumption growth. Since agents dislike the variation of consumption growth, they discount the conditional mean of consumption growth under the risk neutral measure as the compensation for risk. The conditional mean dynamics of state variables also change in a similar reason. In this case, the persistence of X_t can be affected by probability measure transformation. To illustrate this point, suppose that X_t follows an affine process under \mathbb{Q} measure so that $\tilde{\Phi}(X_t) = \tilde{\Phi} X_t$. Let's assume further that volatility is constant and that $f_c(X_t)$ is a quadratic function of X_t . The latter implies that $\Gamma_c(X_t)$ is an affine function of X_t . For example, it can be $\lambda_{c,1} + H_c X_t$. As a result, X_t follows an affine process under \mathbb{P} measure but the persistence of X_t is different from that under \mathbb{Q} measure as shown below:

$$E_t^P(\Delta X_{t+1}) = [\tilde{\Phi} - (\theta - 1) k_{c,2} \Omega(X_t) H_c] X_t - (\theta - 1) k_{c,2} \Omega(X_t) \lambda_c. \quad (28)$$

In this case, preference parameters governing θ and market price of risk parameters H_c can increase or decrease the persistence of X_t under \mathbb{P} measure compared to \mathbb{Q} measure even in the constant volatility case where $\Omega(X_t) = \Omega$.

2.4 Risk Premium Dynamics

Our flexible specification of log price/cash flow ratios implies much richer dynamics of equity risk premium. For instance, the expected excess return on the dividend claim ($r_{d,t+1}^e$) can be time-varying even without time-varying volatility in cash flow dynamics because the market price of risk can be time-varying. In contrast, most long-run risks models that assume the linear approximation to log price/cash flow ratios can generate time-varying risk premium only through time-varying volatility in cash flow dynamics. As Beeler and Campbell (2012) point out, these long-run risks models tend to imply much tighter relations between stock market return and stock market volatility than observed in the U.S. data. Our framework shows that this limitation is not an inherent feature of long-run risk models, but a result of the linear approximation to log price/cash flow ratio.¹² To illustrate this point more clearly, we consider the following expression of the equity risk premium (after ignoring Jensen's inequality term):

$$E_t^P(r_{d,t+1}^e) \equiv E_t^P(r_{d,t+1}) - E_t^Q(r_{d,t+1}) = (E_t^P - E_t^Q) [k_{d,2}\Delta z_{d,t+1} + \Delta d_{t+1}]. \quad (29)$$

Under our assumptions, the above equity risk premium can be decomposed into multiple sources.

$$\begin{aligned} E_t^P(r_{d,t+1}^e) &= (E_t^P - E_t^Q) [k_{d,2}\Gamma_d(X_t)' \Delta X_{t+1} + \Delta d_{t+1}] \\ &= -k_{d,2}k_{c,2}(\theta - 1)\Gamma_d(X_t)' \Omega(X_t)\Gamma_c(X_t) + \gamma\rho_{cd}\sigma_c(X_t)\sigma_d(X_t) \\ &= \frac{\gamma - 1/\psi}{1 - 1/\psi} k_{d,2}k_{c,2}\Gamma_d(X_t)' \Omega(X_t)\Gamma_c(X_t) + \gamma\rho_{cd}\sigma_c(X_t)\sigma_d(X_t). \end{aligned} \quad (30)$$

The equity risk premium consists of two terms. The first term corresponds to the long-run risk channel that shows up only when recursive preferences ($\gamma \neq \frac{1}{\psi}$) are combined with predictable components in cash flows (X_t). The second term corresponds to the standard consumption risk channel that pops out even in the power utility case. Notice that the first term can be time-varying if $\Gamma_d(X_t)$ and $\Gamma_c(X_t)$ are time-varying, even if $\Omega(X_t)$ is constant. Since the linear approximation to log price/cash flow ratios makes $\Gamma_c(X_t)$ and $\Gamma_d(X_t)$ constant, the equity risk premium under that assumption

¹²A similar point was made by Le and Singleton (2010) who argued that constant market price of risk in most long-run risks models was the artifact of the linear approximation to the log price/consumption ratio.

can be time-varying only through time-varying volatility, subjecting models to the criticism raised by Beeler and Campbell (2012). In contrast, our framework allows multiple sources that drive time-varying risk premium and is immune to this criticism.

2.5 Solutions for Steady State Log Price/Cash Flow Ratios

So far, we have treated $k_{c,0}, k_{c,1}, k_{c,2}, k_{d,0}, k_{d,1}$, and $k_{d,2}$ as exogenous. However, they depend on the steady-state values of price-consumption and price-dividend ratios. To obtain the model-consistent approximation of the return process, we have to solve for the steady state value of X_t, \bar{X} , under the physical probability measure. These values can be found by setting $\Phi(\bar{X}) = 0$, where $\Phi(X_t) = E_t^P(\Delta X_{t+1})$. When the number of state variables is n_x , the condition provides n_x nonlinear equations for n_x variables. Once we obtain \bar{X} , we can plug that into the pricing formula to get $\bar{z}_c = f_c(\bar{X})$, which in turn provides values for $k_{c,0}, k_{c,1}$ and $k_{c,2}$. For $k_{d,0}, k_{d,1}$, and $k_{d,2}$, we equate the historical mean of the log price/dividend ratio to the steady state log price/dividend ratio. We also numerically check the uniqueness of solutions. In the subsequent empirical analysis, we exclude parameter values that do not guarantee the existence of the unique solution of the steady-state log price/consumption ratio.

2.6 The Relation to Standard Long-run Risks Models

Standard long-run risks models following Bansal and Yaron (2004) start from specifying physical measure dynamics of cash flows and derive price-cash flow ratios from Euler equations. Therefore, parameters governing the physical measure dynamics of cash flows pin down $f_c(X_t)$ and $f_d(X_t)$ together with preference parameters in the utility function. In contrast, our model treats parameters determining the functional form of $f_c(X_t)$ and $f_d(X_t)$ as free parameters and derive the physical measure dynamics of cash flows consistent with asset pricing restrictions. Indeed, we can nest a typical long-run risk model as a special case of our setup if we impose additional restrictions on model parameters in order to make the physical measure dynamics of cash flows consistent with assumptions in the standard long-run risks model.

For example, consider the one-factor long-run risk model with constant volatility.¹³

¹³The appendix discusses similar restrictions in a two factor long-run risks model with stochastic volatility. In that case, we need to impose additional restrictions on parameters governing stochastic processes of X_t under the risk-neutral measure as well as market prices of risks. The additional restriction is required to interpret one of the risk factors as expected consumption growth uncorrelated to volatility.

Assume that $f_c(X_t)$ and $f_d(X_t)$ are linear with respect to X_t as $\lambda_{c,0} + \lambda_{c,1}X_t$ and $\lambda_{d,0} + \lambda_{d,1}X_t$, respectively. In this case, the only risk factor priced in asset markets is the expected consumption growth factor. By imposing asset pricing restrictions, we obtain the following dynamics of consumption growth and dividend growth.

$$E_t^P(\Delta c_{t+1}) = \text{constant} + \frac{1}{1 - 1/\psi} \left(k_{c,1} - k_{c,2}\tilde{\phi} \right) \lambda_{c,1}X_t, \quad (31)$$

$$E_t^P(\Delta d_{t+1}) = \text{constant} + \frac{1}{\psi - 1} \left(k_{c,1} - k_{c,2}\tilde{\phi} \right) \lambda_{c,1}X_t + \left(k_{d,1} - k_{d,2}\tilde{\phi} \right) \lambda_{d,1}X_t. \quad (32)$$

If we let

$$\lambda_{c,1} = \frac{1 - 1/\psi}{k_{c,1} - k_{c,2}\tilde{\phi}}, \quad \lambda_{d,1} = \frac{q - 1/\psi}{k_{d,1} - k_{d,2}\tilde{\phi}},$$

We then have

$$E_{t+1}^P \Delta c_{t+1} = \text{constant} + X_t, \quad (33)$$

$$E_{t+1}^P \Delta d_{t+1} = \text{constant} + qX_t. \quad (34)$$

In this case, X_t is the expected consumption growth and q is the leverage ratio as specified in Bansal and Yaron (2004). Since $f_c(X_t)$ and $f_d(X_t)$ are affine functions, $\Gamma_c(X_t) = \lambda_{c,1}$ and $\Gamma_d(X_t) = \lambda_{d,1}$ are constants, meaning the market price of risk for X_t is also a constant. In contrast, our model allows nonlinear (for example, quadratic) specifications for $f_c(X_t)$ and $f_d(X_t)$ and incorporate time-varying market price of risk by letting $\Gamma_c(X_t)$ and $\Gamma_d(X_t)$ depend on X_t .

2.7 The Relation to the External Habit Model

Another leading consumption-based asset pricing model is that of Campbell and Cochrane (1999) which features an external habit in consumption. In contrast to the long-run risks model, the habit model assumes i.i.d. growth rate of cash flow and attributes all volatilities in asset price to variations in expected returns. Risk premia are time-varying because of variations in the representative consumer's risk aversion as her consumption fluctuates around a slow-moving habit level. In particular, the model assumes that the log stochastic discount factor is

$$m_{t+1} = \log \beta - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1}, \quad (35)$$

where s_t is the log surplus consumption ratio that measures the deviation of consumption from an external habit. s_t is assumed to have the following law of motion (under the physical probability measure):

$$s_{t+1} - \bar{s} = \rho(s_t - \bar{s}) + \lambda_t(\Delta c_{t+1} - \mu_c), \quad (36)$$

and

$$\Delta c_{t+1} = \mu_c + \sigma_c \varepsilon_{t+1}, \quad (37)$$

where ε_{t+1} is i.i.d. standard normal and is the only shock in the model. The key element in the model is the “sensitivity” function λ_t that governs the conditional covariance between consumption growth and the surplus-consumption ratio. λ_t is assumed to be a function of s_t and produces time-varying risk premia in the model.

We show below that, by imposing additional restrictions on the market prices of risk, we can also obtain a version of our model with i.i.d. consumption growth and time-varying risk premia. As in the habit model, we assume there is only one shock in the model and consider a single-factor quadratic model of asset prices where log wealth-consumption ratio is given by

$$f_c(x_t) = \lambda_0 + \lambda_1 x_t + \frac{1}{2} \lambda_2 x_t^2 \quad (38)$$

and

$$\Gamma_c(x_t) = \frac{\partial f_c(x_t)}{\partial x_t} = \lambda_1 + \lambda_2 x_t \quad (39)$$

where $\lambda_2 \neq 0$. We assume that, under the risk-neutral probability measure, the laws of motion for consumption, c_t , and the state variable, x_t , are given by:

$$\Delta c_{t+1} = \tilde{g}_c(x_t) + \sigma_c \tilde{\varepsilon}_{t+1} \quad (40)$$

$$\Delta x_{t+1} = \tilde{\phi} x_t + \sigma_x \tilde{\varepsilon}_{t+1} \quad (41)$$

where $\tilde{\varepsilon}_{t+1}$ is an i.i.d. shock with standard normal distribution under the risk neutral probability measure.

As in (21), the equilibrium condition under the recursive utility function then implies

that

$$\begin{aligned}\tilde{g}_c(x_t) = & -\frac{\theta}{1-\gamma}[\ln \delta + k_{c,0} - k_{c,1}f_c(x_t) + k_{c,2}\Gamma_c(x_t)\tilde{\phi}x_t] \\ & + \frac{1}{2}[(1-\theta)k_{c,2}\Gamma_c(x_t)\sigma_x + \gamma\sigma_c]^2 - \frac{1}{2}[k_{c,2}\Gamma_c(x_t)\sigma_x + \sigma_c]^2\end{aligned}\quad (42)$$

It then follows that, under the physical probability measure, the law of motion for consumption growth is:

$$\Delta c_{t+1} = \tilde{g}_c(x_t) + [(1-\theta)k_{c,2}\Gamma_c(x_t)\sigma_x\sigma_c + \gamma\sigma_c^2] + \sigma_c\varepsilon_{t+1}\quad (43)$$

where ε_{t+1} is i.i.d. normal under the physical probability measure.

It is then easy to show that,

$$E_t^P(\Delta c_{t+1}) = \text{constant}\quad (44)$$

under the following restrictions on λ_1 and λ_2 :

$$\lambda_1 = -\frac{[(1-\theta)\gamma - 1](1-\gamma)}{\theta\tilde{\phi}}\sigma_x\sigma_c\lambda_2\quad (45)$$

$$\lambda_2 = \frac{k_{c,1} - 2k_{c,2}\tilde{\phi}}{(1-\gamma)(2-\theta)k_{c,2}^2\sigma_x^2}\quad (46)$$

While consumption growth is i.i.d., the risk premium on the consumption claim, however, is time-varying because of time-varying market price of risk:

$$(E_t^P - E_t^Q)r_{c,t+1} = [k_{c,2}\Gamma_c(x_t)\sigma_x/\sigma_c + 1](E_t^P - E_t^Q)\Delta c_{t+1}\quad (47)$$

or

$$(E_t^P - E_t^Q)r_{c,t+1} = [(1-\theta) + \gamma]k_{c,2}\Gamma_c(x_t)\sigma_x\sigma_c + (1-\theta)k_{c,2}^2\Gamma_c^2(x_t)\sigma_x^2 + \gamma\sigma_c^2\quad (48)$$

In the habit model of Campbell and Cochrane (1999), x_t is introduced into the model through the utility function with the economic interpretation of consumption habit. In our model, x_t can be interpreted as an exogenous risk premium shock that is perfectly correlated with the consumption shock.¹⁴

¹⁴The model changes very little if we assume innovations to Δx_{t+1} is imperfectly correlated with

3 Data and Estimation Methodology

3.1 Data

We use the annual stock price and dividend data for S&P index as well as 1 year nominal yield data available on Robert Shiller's website(<http://www.econ.yale.edu/shiller/data.htm>). We construct ex-ante real interest rate by subtracting expected CPI inflation from ARMA (1,1) model from the 1 year nominal bond yield data. Real per capital consumption growth is also available on the same website. For the estimation of an equilibrium model, we use data from 1929 to 2013. Figure 1 provides the time series plots of data used in the estimation.

3.2 Econometric Methodology

Once we have the joint pdf for $(\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1} | X_t)$, the joint pdf of $(\Delta c_{t+1}, \Delta d_{t+1}, r_{d,t+1} - \Delta d_{t+1}, r_{f,t} | X_t)$ can be easily obtained by a change of variable using the delta method. In particular, if we let $Y_t = (\Delta c_{t+1}, \Delta d_{t+1}, r_{d,t+1} - \Delta d_{t+1}, r_{f,t})'$, we can have a (nonlinear) state space representation of cash flows and asset returns under \mathbb{P} as follows:¹⁵

$$Y_{t+1} = Y(X_t, \epsilon_{c,t+1}, \epsilon_{d,t+1}, \eta_{t+1}, u_{f,t}), \quad (49)$$

$$X_t = X_{t-1} + \Phi(X_{t-1}) + \eta_t, \quad (50)$$

where $u_{f,t}$ is a normal random variable to capture a measurement error in the real risk-free rate.¹⁶

If we take the linear approximation to the price-cash flow ratio and assume constant volatility and the affine process for X_t , the above model becomes a linear state space model. In this case, we can use Kalman filter to back out latent X_t from observed data. However, if not all these assumptions hold, the model becomes a nonlinear state space model and Kalman filter is no longer optimal to back out X_t . In this case, we can use

innovations to Δc_{t+1} .

¹⁵We consider $r_{d,t+1} - \Delta d_{t+1}$ rather than $r_{d,t+1}$ because the independence between η_{t+1} and $(\epsilon_{c,t+1}, \epsilon_{d,t+1})$ makes the implementation of particle filtering computationally more efficient. This set-up assures that the evaluation of probability density for all the particles can be vectorized, which speeds up our computation in **MATLAB** substantially.

¹⁶We set the standard deviation of $u_{f,t}$ equal to the standard deviation of $r_{f,t}$. In our implementation of particle filtering, the standard deviation of measurement error plays like a bandwidth parameter in a nonparametric density estimation. This relatively high value allows more particles to be used in the calculation of the likelihood, resulting in the smooth approximation of likelihood.

simulation based particle filtering to back out X_t and calculate the associated likelihood.

Let ϑ be a vector of model parameters to be estimated. ϑ consists of model parameters determining the state space representation. Once we construct the likelihood of Y_t under \mathbb{P} measure, $p(Y_t|\vartheta)$, we can obtain the maximum likelihood estimate of ϑ by simply maximizing $p(Y|\vartheta)$ over the parameter space. In this paper, we take the Bayesian approach and add prior information about ϑ , $p(\vartheta)$, into the likelihood and obtain posterior draws of ϑ . Bayesian methods can better characterize the uncertainty related to model parameters conditional on relatively short sample observations.¹⁷ As in Schorfheide et al. (2014), we apply Markov Chain Monte Carlo (MCMC) methods to obtain posterior draws of ϑ . Although these draws are generated by Markov Chain that allows serial correlations between draws, under certain regularity conditions, these draws behave like independently distributed draws from the stationary distribution $p(\vartheta|Y)$. Once we obtain posterior draws of ϑ , we can perform posterior simulation of X_t to compute moments of interest including cash flow and stock market return predictive regression coefficients.¹⁸

In estimation, we calibrate two constant parameters in $f_c(X_t)$ and $f_d(X_t)$ ($\lambda_{c,0}$ and $\lambda_{d,0}$) conditional on other model parameters to match the average consumption growth and log price-dividend ratio with the model implied unconditional means. They are difficult to identify from the likelihood because unconditional means of observed variables under \mathbb{P} measure are highly nonlinear functions due to asset pricing restrictions.¹⁹

4 Empirical Analysis

4.1 Model Specification and Prior Distribution

In empirical analysis, we consider a two factor long-run risks model in which time varying expected consumption growth and time-varying consumption volatility are two risk factors. We apply the following quadratic approximation to log price/cash flow ratios.

¹⁷Our sample covers a long period of time, but the number of observations is relatively short because data are available at the low frequency.

¹⁸We use the likelihood function approximated by particle filters inside the Metropolis-Hastings algorithm that is used to generate posterior draws of parameters. For details of this algorithm, see the Chapter 9 of Herbst and Schorfheide (2015).

¹⁹In particular, second moments of X_t are highly nonlinear functions of model parameters.

$$f_c(X_t) = \lambda_{c,0} + \lambda'_{c,1}X_t + \frac{1}{2}X_t'H_cX_t, \quad (51)$$

$$f_d(X_t) = \lambda_{d,0} + \lambda'_{d,1}X_t + \frac{1}{2}X_t'H_dX_t, \quad (52)$$

where, for $i = c, d$,

$$\lambda'_{i,1} = \begin{pmatrix} \lambda_{i,11} \\ \lambda_{i,12} \end{pmatrix}, \quad H_i = \begin{pmatrix} h_{i,11} & 0 \\ 0 & h_{i,22} \end{pmatrix}$$

and hence

$$\Gamma_c(X_t) = \lambda_{c,1} + H_cX_t \quad (53)$$

$$\Gamma_d(X_t) = \lambda_{d,1} + H_dX_t \quad (54)$$

Dynamics of X_t are given by the following VAR(1) process under \mathbb{Q} :

$$\begin{aligned} \Delta X_{t+1} &= \tilde{\Phi}X_t + \Sigma(X_t)\tilde{\eta}_{t+1} \\ &= \begin{pmatrix} \tilde{\phi}_{11} & \tilde{\phi}_{12} \\ 0 & \tilde{\phi}_{22} \end{pmatrix} X_t + \begin{pmatrix} \phi_x(\sqrt{\alpha + x_{2,t}}) & 0 \\ 0 & \sigma_2 \end{pmatrix} \tilde{\eta}_{t+1} \end{aligned} \quad (55)$$

where α is non-negative, and $\tilde{\eta}_{t+1}$ is a 2×1 i.i.d. standard normal random vector under \mathbb{Q} . The volatility of unexpected consumption growth is $\sigma_c(X_t) = \sqrt{\alpha + x_{2,t}}$ and the volatility of unexpected dividend growth is $\sigma_d(X_t) = \phi_d\sqrt{\alpha + x_{2,t}}$. Since the volatility factor $x_{2,t}$ is assumed to follow an AR(1) process independently of $x_{1,t}$, we can clearly distinguish the volatility factor from the non-volatility factor.

To elicit prior distributions of parameters, we start from calibrated values of parameters in Bansal and Yaron (2004). In their setup, market price of risk parameters ($\lambda_{c,1}$ and $\lambda_{d,1}$) are nonlinear functions of other model parameters. So we can generate prior distribution of $\lambda_{c,1}$ and $\lambda_{d,1}$. For quadratic terms of log price/cash flow ratios, we use diffuse priors centered around zero. For other parameters, we draw on Schorfheide et al. (2014) and Bansal and Yaron (2004) to set prior distributions. Table 1 summarizes our prior specification of model parameters.

4.2 Posterior Estimates

Table 2 shows posterior distribution of model parameters. In addition to the baseline model in which log price/cash flow ratios are assumed to be quadratic functions of state variables, we also estimate the nested model that log price/cash flow ratios are restricted to be linear functions of state variables. In both models, posterior distributions imply that state variables are highly persistent. In standard long-run risks models, preference parameters are calibrated to imply that agents prefer the early resolution of uncertainty ($\gamma > \frac{1}{\psi}$) and a positive shock to expected consumption ratio increases the log price-consumption ratio through a strong intertemporal substitution channel ($\psi > 1$). Under that combination of model parameters, $\lambda_{c,11} > 0$ and $\lambda_{c,12} < 0$. While our posterior mode estimates of γ and ψ satisfy such restrictions, $\lambda_{c,12} > 0$. This occurs in our model because the risk-free rate declines substantially in response to a positive volatility shock. Therefore, the consumption claim that promises to provide the same payoff stream in the future is valued more highly. Posterior distributions of $h_{c,11}$ and $h_{d,11}$ deviate significantly from prior distributions centered around zero while the difference between prior and posterior distributions is relatively minor in case of $h_{c,22}$ and $h_{d,22}$. The finding suggests that time-varying market prices of risks are mostly driven by shocks to expected cash flow factor ($x_{1,t}$).²⁰ At the posterior mode, $h_{c,11} < 0$ and $h_{d,11} > 0$. The semi-elasticity of price/consumption ratio is decreasing in $x_{1,t}$ while the semi-elasticity of price/dividend ratio is increasing in $x_{1,t}$.²¹ This suggests that the constant-leverage-ratio assumption maintained in most long-run risk models may be too restrictive and not consistent with data.

The general expression of the equity premium is given in (30). The part of the equity premium associated with $x_{1,t}$ in our quadratic model is given by:

$$\frac{\gamma - 1/\psi}{1 - 1/\psi} k_{d,2} k_{c,2} \{ (h_{c,11} \lambda_{d,11} + h_{d,11} \lambda_{c,11}) x_{1,t} + h_{c,11} h_{d,11} x_{1,t}^2 \} \sigma_{1,t}^2$$

where $\sigma_{1,t}^2$ is the conditional variance of $x_{1,t+1}$ which depends only on the volatility factor $x_{2,t}$. We can easily verify that, at the posterior mode of model parameters, holding the quantity of risk, $\sigma_{1,t}^2$, constant, this term is decreasing in $x_{1,t}$. The equity premium

²⁰Although the posterior mode estimate of $h_{d,22}$ is substantially positive, the equity risk premium is determined by $\Gamma_c(X_t)\Omega(X_t)\Gamma_d(X_t)$ and the low value of $h_{c,22}$ mitigates the role of time-varying market price of volatility risk in driving equity risk premium.

²¹Note that $\frac{\partial f_c(X_t)}{\partial x_{1,t}} = \lambda_{c,11} + h_{c,11} x_{1,t}$, $\frac{\partial f_d(X_t)}{\partial x_{1,t}} = \lambda_{d,11} + h_{d,11} x_{1,t}$, where $f_c(X_t)$ and $f_d(X_t)$ are log price/consumption ratio and log price/dividend ratio respectively.

is counter-cyclical even in the absence of time-varying volatility (see more discussions below).

The posterior estimate of the relative volatility of a persistent shock to cash flows (ϕ_x) is much higher than the typical value reported in the long-run risks literature. For instance, the posterior 90% interval of the corresponding parameter in Schorfheide et al. (2014) is [0.03, 0.04] while our estimate is [0.20, 0.47]. In standard long-run risks model, cash flows become much more predictable, the higher this parameter value is. Jacking up this value in the calibrated one-factor long-run risks model of Bansal and Yaron (2004) to our posterior estimate implies that the first-order autocorrelation of consumption growth is twice as high as in the sample.²² This does not happen in our model because the expected cash flow factor, $x_{1,t}$ can be much less persistent under the physical probability measure than that under the risk-neutral probability measure because of time-varying market prices of risks that depends on $x_{1,t}$. In fact, from (27), we can see that in our quadratic model,

$$E_t^P(x_{1,t+1}) = \left\{ \tilde{\phi}_{11} - (\theta - 1)k_{c,2}h_{c,11}\sigma_{1,t}^2 \right\} x_{1,t} + \dots$$

where $\sigma_{1,t}^2$ is again the conditional variance of $x_{1,t+1}$. At the posterior mean, $\tilde{\phi}_{11}$ is -0.014, implying a very persistent expected cash flow process under the risk probability measure. But since $\theta < 0$ and $h_{c,11} < 0$, the persistence of $x_{1,t}$ can be greatly reduced in the physical probability measure.²³ This feature of the quadratic model helps reconcile long-run risk models with the data on return and cash flow predictability. To see this point more clearly, we turn to posterior predictive analysis.

4.3 Posterior Predictive Analysis

In this section, we examine how well our model fits sample moments in cash flows and asset returns data by using posterior predictive analysis. As in Schorfheide et. al. (2014), we simulate data from our posterior parameter estimates and then calculate moments from each simulated dataset. We compare distribution of these predictive moments with sample moments from the data. Table 3 shows moments of cash flows and asset returns that we try to match. Except for volatilities of stock market return and the log price/dividend ratio, both linear and quadratic models match first and second moments

²²In the sample data, the autocorrelation of consumption growth 0.31 while the model-implied persistence of consumption growth becomes about 0.6 when we increase ϕ_x to 0.3.

²³Note that both $k_{c,2}$ and $\sigma_{1,t}^2$ are positive.

of cash flows and asset returns relatively well in the sense that sample moments from the data are included in the posterior 90% interval. The moment match is relatively poor for correlation coefficients. Nonetheless, except for the autocorrelation of the risk-free rate, the sample moment is not that far from posterior predictive distributions.

One of the main criticisms against long-run risks models highlighted by Beeler and Campbell (2012) is the high predictability of cash flows by the log price/dividend ratio implied by the model. To address this issue, we consider following regressions:

$$\begin{aligned} \sum_{h=1}^H \Delta c_{t+h} &= c_0 + c_1 z_{d,t} + resid_{c,t+H}, \\ \sum_{h=1}^H \Delta d_{t+h} &= d_0 + d_1 z_{d,t} + resid_{d,t+H}, \\ \sum_{h=1}^H (r_{d,t+h} - r_{f,t+h-1}) &= h_0 + h_1 z_{d,t} + resid_{r,t+H}. \end{aligned} \tag{56}$$

In the data, the log price/dividend ratio predicts excess market return especially at a longer horizon as Table 4 shows. The linear model with long-run risks restrictions implies much higher predictability of cash flows at a longer horizon than in the data. It also implies much smaller predictability of excess market return. While the linear model without long-run risks restrictions improves the predictability of cash flows, it still generates much smaller predictability of excess market return than in data. Overall, our quadratic model does better in both dimensions. In particular, the log price/dividend ratio strongly predicts excess market return than cash flows at a long horizon to be consistent with data.

4.4 Estimates of Expected Cash Flows and Risk Premium

Since our model determines cash flow dynamics under the physical probability measure by using asset pricing restrictions, the model can generate counterfactual cash flow dynamics if asset pricing restrictions are misspecified. To look into this issue, we generate model-implied expectations of cash flows at the posterior mode estimates of the quadratic model. Figures 2 and 3 show time-series plots of expected cash flows with realized data. While the volatilities of realized cash flows are dominated by unexpected innovations, the cyclical pattern of model-implied expectations looks reasonable. During most recession periods identified by the National Bureau of Economic Research (NBER), expectations

of cash flows decline and recover at the end of recession periods.

Assured that implications of cash flow dynamics are reasonable, we now examine the model-implied risk premium dynamics. Beeler and Campbell (2012) argue that long-run risks models along the line of Bansal and Yaron (2004) imply that the relation between stock market volatility and stock price is too tight. The fundamental reason for this is that the market price of risk is constant and the variation of risk premium comes only through time-varying volatility of cash flows. Our quadratic model relaxes this restrictions by allowing the market price of risk to be time-varying and not proportional to volatility. To see if the model is successful in generating reasonable estimates for both volatility and risk premium, we turn to model-implied estimates of consumption volatility and equity risk premium.

First, Figure 4 shows the model-implied estimates of volatility of consumption growth at the posterior mode of the quadratic model. Again, during most recessionary periods identified by the NBER, our consumption volatility increases at the beginning and starts to decline around the end of recessions. It is noticeable that consumption volatility jumped up substantially during the recession of 2007-9. Overall, our volatility estimates are counter-cyclical which are consistent with the literature emphasizing the recessionary force of an uncertainty shock (Bloom (2009), for example).

Second, we turn to the model-implied estimates of equity risk premium to see if the model can generate a reasonable counter-cyclical pattern of risk premium. We compute the equity risk premium in terms of the expected excess return as explained in the previous discussion. Figure 5 shows two estimates of equity risk premium. The solid line shows the time-series plot of the equity risk premium at the posterior mode of the quadratic model. It is clear that the model-implied equity risk premium estimates show a counter-cyclical pattern as much noted in the literature.

To highlight the role of the time-varying market price of risk, we plot another risk premium estimates purely through time-varying volatility in the dash-dot line. These alternative estimates of the equity risk premium quite closely follow the volatility estimates because the equity risk premium is linear with respect to volatility in this case. These alternative estimates are more volatile than the realized excess market return, which is impossible in theory. The realized excess market return is the sum of the expected excess return and the unexpected component. Since the two components are orthogonal by definition, the volatility of the realized excess market return is also the sum of the volatility of each component. In the data, the volatility of the realized ex-

cess market return is 19% while the volatility of alternative estimates of the expected excess return is 22%. In contrast, the volatility of the expected excess return with the time-varying market price of risk is 8%. Our findings suggest that the quadratic model provides reasonable implications for both cash flow and risk premium dynamics.

5 Conclusions

We develop and estimate an equilibrium asset pricing model with recursive preferences in which the market prices of risk are time-varying. Based on insight from Le and Singleton (2010), we use a quadratic approximation to the log price/consumption ratio to derive time-varying market prices of risk from a preference-based model. We extend their approach to a setting with multiple assets and cash flows to identify and estimate consumption risk priced in the aggregate stock market. We use the long-run historical data on cash flows and asset returns in the U.S. and estimate the model using Bayesian methods. By doing so, we try to overcome two counterfactual implications of long-run risks models following Bansal and Yaron (2004): the over-predictability of cash flows by asset returns and the tight relation between risk premium and return volatility. We argue that both criticisms can be attributed to the assumption of the constant market prices of risk and restrictive econometric specifications for the expected cash flow processes in the existing literature. Once we relax these assumptions, the log price/dividend ratio predicts excess market return more strongly than cash flows especially at a longer horizon as in the data. Also, the estimated model generates counter-cyclical risk premium that is of a plausible magnitude and is not proportional to consumption volatility. Our results suggests that a flexible specification of the market prices of risk in an otherwise standard long-run risks model goes a long way to reconciling model implied cash flow and risk premium dynamics with the historical data.

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A Linear Model

Here, we consider a restricted version of our model in which log price/cash flow ratios are assumed to be linear. Let $X_t = (x_{1,t}, x_{2,t})'$. Log price/cash flow ratios are as follows:

$$\begin{aligned} f(X_t) &= \lambda_0 + \lambda' X_t \\ &= \lambda_0 + \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix}' X_t \end{aligned} \quad (57)$$

where X_t follows the same dynamics as our quadratic model described in the text.

Apply the model to claims on consumption and dividend streams,

$$f_c(X_t) = \lambda_{c,0} + \lambda'_c X_t, \quad (58)$$

$$f_d(X_t) = \lambda_{d,0} + \lambda'_d X_t. \quad (59)$$

where, for $i = c, d$,

$$\lambda'_i = \begin{pmatrix} \lambda_{i,1} \\ \lambda_{i,2} \end{pmatrix},$$

and hence

$$\Gamma_c = \lambda_c, \quad \Gamma_d = \lambda_d. \quad (60)$$

We further assume that, under \mathbb{Q} ,

$$\Delta c_{t+1} = \tilde{g}_c(X_t) + \sqrt{\alpha + x_{2,t}} \tilde{\varepsilon}_{c,t+1} \quad (61)$$

$$\Delta d_{t+1} = \tilde{g}_d(X_t) + \phi_d \sqrt{\alpha + x_{2,t}} \tilde{\varepsilon}_{d,t+1} \quad (62)$$

where for $i = c, d$, $\sigma_i > 0$, and $\tilde{\varepsilon}_{i,t+1}$ is standard normal under \mathbb{Q} . We denote the correlation coefficient between $\tilde{\varepsilon}_{c,t+1}$ and $\tilde{\varepsilon}_{d,t+1}$ as ρ_{cd} . We assume $\tilde{\varepsilon}_{i,t+1}$ is independent of $\tilde{\eta}_{t+1}$.

Equilibrium asset pricing restrictions under the recursive utility function imply:

$$\begin{aligned} \tilde{g}_c(X_t) &= -\frac{\theta}{1-\gamma} [\ln \delta + k_{c,0} - k_{c,1} f_c(X_t) + k_{c,2} \Gamma'_c \tilde{\Phi} X_t] \\ &\quad - \frac{\theta(2-\theta)}{2(1-\gamma)} k_{c,2}^2 \Gamma'_c \Omega(X_t) \Gamma_c - \frac{1+\gamma}{2} (\alpha + \beta x_{2,t}) \end{aligned} \quad (63)$$

and

$$\begin{aligned}
\tilde{g}_d(X_t) - \tilde{g}_c(X_t) &= (k_{c,0} - k_{d,0}) - [k_{c,1}f_c(X_t) - k_{d,1}f_d(X_t)] \\
&\quad + [k_{c,2}\Gamma_c - k_{d,2}\Gamma_d]' \tilde{\Phi} X_t + \frac{1}{2}(1 - \phi_d^2)(\alpha + x_{2,t}) \\
&\quad + \frac{1}{2}[k_{c,2}^2\Gamma'_c\Omega(X_t)\Gamma_c - k_{d,2}^2\Gamma'_d\Omega(X_t)\Gamma_d]
\end{aligned} \tag{64}$$

Under the physical probability measure \mathbb{P} ,

$$E_t^P(\Delta X_{t+1}) = \tilde{\Phi} X_t - (\theta - 1)k_{c,2}\Omega(X_t)\Gamma_c \tag{65}$$

$$E_t^P(\Delta c_{t+1}) = \tilde{g}_c(X_t) + \gamma(\alpha + x_{2,t}) \tag{66}$$

$$E_t^P(\Delta d_{t+1}) = \tilde{g}_d(X_t) + \gamma\rho_{cd}\phi_d(\alpha + x_{2,t}) \tag{67}$$

and

$$\Delta X_{t+1} - E_t^P(\Delta X_{t+1}) = \Sigma(X_t)\eta_{t+1} \tag{68}$$

$$\Delta c_{t+1} - E_t^P(\Delta c_{t+1}) = \sqrt{\alpha + x_{2,t}}\varepsilon_{c,t+1} \tag{69}$$

$$\Delta d_{t+1} - E_t^P(\Delta d_{t+1}) = \phi_d\sqrt{\alpha + x_{2,t}}\varepsilon_{d,t+1} \tag{70}$$

where

$$\begin{pmatrix} \eta_{t+1} \\ \varepsilon_{c,t+1} \\ \varepsilon_{d,t+1} \end{pmatrix} \sim \mathcal{N} \left(0, \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & \rho_{cd} \\ 0 & \rho_{cd} & 1 \end{pmatrix} \right)$$

B Relation to the Two Factor Long-run Risks Model

The standard long-run risks model in which $x_{1,t}$ corresponds to fluctuations in expected consumption growth and $x_{2,t}$ is the volatility of the unexpected consumption growth can be obtained as a special case of the linear model described in the previous section. Recall that the expected consumption growth in the linear model is given by

$$\begin{aligned}
E_t^P(\Delta c_{t+1}) &= \text{constant} + \frac{\theta k_{c,1}}{1 - \gamma} \lambda'_c X_t - \frac{\theta k_{c,2}}{1 - \gamma} \lambda'_c \tilde{\Phi} X_t \\
&\quad - \frac{\theta(2 - \theta)k_{c,2}^2}{2(1 - \gamma)} \lambda_{c,11}^2 x_{2,t} - \frac{1 - \gamma}{2} x_{2,t}.
\end{aligned} \tag{71}$$

In the standard long-run risk model, it is assumed that $E_t^P(\Delta c_{t+1}) = x_{1,t}$, this is equivalent to imposing the following restriction on λ_c :

$$\frac{\theta}{1-\gamma}(k_{c,1}I - k_{c,2}\tilde{\Phi}')\lambda_c - \frac{\theta(2-\theta)k_{c,2}^2}{2(1-\gamma)}(0, \lambda_{c,11}^2)' - \frac{1-\gamma}{2}(0, 1)' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (72)$$

where I is a 2×2 identity matrix.

Similarly, the expected dividend growth is given by

$$\begin{aligned} E_t^P(\Delta d_{t+1}) &= \text{constant} + E_t^P(\Delta c_{t+1}) - \gamma x_{2,t} \\ &\quad - (k_{c,1}\lambda_c - k_{d,1}\lambda_d)'X_t + (k_{c,2}\lambda_c - k_{d,2}\lambda_d)'\tilde{\Phi}X_t \\ &\quad + \frac{1}{2}(1 - \phi_d^2)(0, x_{2,t})' + \frac{1}{2}(k_{c,2}^2\lambda_{c,11}^2 - k_{d,2}^2\lambda_{d,11}^2)(0, x_{2,t})' \\ &\quad + \gamma\rho_{cd}\phi_d(0, x_{2,t})'. \end{aligned} \quad (73)$$

The standard long-run risk model assumes that

$$E_t^P(\Delta d_{t+1}) = qE_t^P(\Delta c_{t+1})$$

where q is constant representing the leverage ratio.

This is equivalent to imposing the following restrictions on λ_d ,

$$\begin{aligned} \begin{pmatrix} q-1 \\ 0 \end{pmatrix} &= \left[-\gamma + \frac{1}{2}(1 - \phi_d^2) + \gamma\rho_{cd}\phi_d \right] (0, 1)' \\ &\quad + \left[-(k_{c,1}\lambda_c - k_{d,1}\lambda_d) + \tilde{\Phi}'(k_{c,2}\lambda_c - k_{d,2}\lambda_d) \right] \\ &\quad + \frac{1}{2}(k_{c,2}^2\lambda_{c,11}^2 - k_{d,2}^2\lambda_{d,11}^2)(01). \end{aligned} \quad (74)$$

Finally, two state variables are independent under the physical probability measure in the standard long-run risks model. This restricts $\tilde{p}h_{12}$ through the following equation

$$E_t^P(\Delta X_{t+1}) = \tilde{\Phi}X_t - (\theta - 1)k_{c,2}\Omega(X_t)\Gamma_c = \text{constant} + \begin{pmatrix} \tilde{\phi}_{11} & 0 \\ 0 & \tilde{\phi}_{22} \end{pmatrix} X_t. \quad (75)$$

More specifically, $\tilde{\phi}_{12}$ is restricted to be $(\theta - 1)k_{c,2}\lambda_{c,11}\phi_x^2$.

C The Risk-free Rate

We can use the fact $e^{r_{f,t}} = E_t^Q(e^{r_{c,t+1}})$ to obtain the model-implied real risk-free rate:

$$\begin{aligned} r_{f,t} = & k_{c,0} - k_{c,1}f_c(X_t) + k_{c,2}\Gamma_c(X_t)'\tilde{\Phi}(X_t) + \tilde{g}_c(X_t) \\ & + \frac{1}{2}k_{c,2}^2\Gamma_c(X_t)'\Omega(X_t)\Gamma_c(X_t) + \frac{1}{2}\sigma_c^2(X_t) \end{aligned} \quad (76)$$

Using the solution for $\tilde{g}_c(X_t)$ in (21), we can solve for the risk-free rate as $r_{f,t} = h(X_t)$, where

$$\begin{aligned} h(X_t) = & -\frac{\theta}{1-\gamma}\ln\delta + \left(1 - \frac{\theta}{1-\gamma}\right) \left[k_{c,0} - k_{c,1}f_c(X_t) + k_{c,2}\Gamma_c(X_t)'\tilde{\Phi}(X_t) \right] \\ & + \frac{1}{2} \left(1 - \frac{\theta(2-\theta)}{1-\gamma}\right) k_{c,2}^2\Gamma_c(X_t)'\Omega(X_t)\Gamma_c(X_t) - \frac{\gamma}{2}\sigma_c^2(X_t) \end{aligned} \quad (77)$$

Therefore we can add $r_{f,t+1}$ to the observation equation of the non-linear state-space model in (49):

$$r_{f,t} = h(X_t) + u_{f,t} \quad (78)$$

where $u_{f,t}$ is an i.i.d. measurement error for the risk-free rate.

D Likelihood Evaluation

In this appendix we explain the detailed derivations of the joint likelihood function of cash flows and asset returns.

We let $f^P(\cdot|X_t)$ and $f^Q(\cdot|X_t)$ denote the conditional pdf of a random variable under physical and risk-neutral probability measures respectively. It then follows that

$$f^P(\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1}|X_t) = \frac{e^{-m_{t+1}}}{E_t^Q(e^{-m_{t+1}})} f^Q(\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1}|X_t) \quad (79)$$

where

$$\begin{aligned} \frac{e^{-m_{t+1}}}{E_t^Q(e^{-m_{t+1}})} = & e^{-\frac{1}{2}(\theta-1)^2 k_{c,2}^2 \Gamma_c(X_t)' \Omega(X_t) \Gamma_c(X_t) - \frac{1}{2} \gamma^2 \sigma_c^2(X_t)} \\ & \times e^{-(\theta-1) k_{c,2} \Gamma_c(X_t)' \Sigma(X_t) \tilde{\eta}_{t+1} + \gamma \sigma_c(X_t) \tilde{\varepsilon}_{c,t+1}} \end{aligned} \quad (80)$$

By the assumptions in (15), (16) and (17), $f^Q(\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1} | X_t)$ is multivariate normal under the risk-neutral probability measure with conditional mean and conditional variance given by, respectively,

$$\tilde{H}(X_t) = \begin{pmatrix} \tilde{g}_c(X_t) \\ \tilde{g}_d(X_t) \\ \tilde{\Phi}(X_t) \end{pmatrix} \quad (81)$$

and

$$\Xi(X_t) = \begin{pmatrix} \sigma_c^2(X_t) & \rho_{cd}\sigma_c(X_t)\sigma_d(X_t) & 0 \\ \rho_{cd}\sigma_c(X_t)\sigma_d(X_t) & \sigma_d^2(X_t) & 0 \\ 0 & 0 & \Omega(X_t) \end{pmatrix} = \begin{pmatrix} \Xi_1(X_t) & 0 \\ 0 & \Omega(X_t) \end{pmatrix} \quad (82)$$

By (80), the joint conditional pdf of $(\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1})'$ under the physical probability measure is also multivariate normal with the same conditional variance $\Xi(X_t)$ and a conditional mean given by

$$H(X_t) = \tilde{H}(X_t) + \Xi(X_t)\Lambda(X_t) \quad (83)$$

where

$$\Lambda(X_t) = \begin{pmatrix} \gamma \\ 0 \\ -(\theta - 1)k_{c,2}\Gamma_c(X_t) \end{pmatrix} \quad (84)$$

Let $Y_{X,t} = (\Delta c_{t+1}, \Delta d_{t+1}, \Delta X_{t+1})'$, we then have:

$$f^P(Y_{X,t+1} | X_t) = \frac{1}{\sqrt{2\pi}|\Xi(X_t)|^{1/2}} e^{-\frac{1}{2}(Y_{X,t+1} - H(X_t))'\Xi(X_t)^{-1}(Y_{X,t+1} - H(X_t))} \quad (85)$$

Notice that the marginal distribution of X_{t+1} is given by

$$f_X^P(X_{t+1} | X_t) = \frac{1}{\sqrt{2\pi}|\Omega(X_t)|^{1/2}} e^{-\frac{1}{2}(X_{t+1} - X_t - \Phi(X_t))'\Omega(X_t)^{-1}(X_{t+1} - X_t - \Phi(X_t))} \quad (86)$$

where

$$\Phi(X_t) = \tilde{\Phi}(X_t) - (\theta - 1)k_{c,2}\Omega(X_t)\Gamma_c(X_t)$$

If we let $Y_{t+1} = (\Delta c_{t+1}, \Delta d_{t+1}, r_{d,t+1} - \Delta d_{t+1}, r_{f,t})' = (Y'_{1,t}, r_{d,t+1} - \Delta d_{t+1}, r_{f,t})'$, the

model can then be casted in a nonlinear state space form as follows:

$$Y_{1,t+1} = A_1(X_t) + B_1(X_t)H_1(X_t) + \epsilon_{t+1}, \quad (87)$$

$$r_{d,t+1} - \Delta d_{t+1} = k_{d,0} - k_{d,1}f_d(X_t) + k_{d,2}\Gamma_d(X_t)'(\Phi(X_t) + \eta_{t+1}), \quad (88)$$

$$r_{f,t} = h(X_t) + u_{f,t}, \quad (89)$$

$$X_t = X_{t-1} + \Phi(X_{t-1}) + \eta_t. \quad (90)$$

where²⁴

$$A_1(X_t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B_1(X_t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$f^P(\epsilon_{t+1}|X_t) = \frac{1}{\sqrt{2\pi}|B_1(X_t)\Xi_1(X_t)B_1(X_t)'|^{1/2}} e^{-\frac{1}{2}u_{1,t+1}'B_1(X_t)\Xi_1(X_t)B_1(X_t)'u_{1,t+1}} \quad (91)$$

Since η_{t+1} and $u_{f,t}$ are *i.i.d.* normal random variables independent of X_t , it is straightforward to write down the pdf of observed variables $p(r_{d,t+1} - \Delta d_{t+1}, r_{f,t}|X_t)$.

²⁴ $H(X_t)$ is given in (83)

Table 1: PRIOR DISTRIBUTION

Parameters	Domain	Density	Para(1)	Para(2)
$\lambda_{c,11}$	\mathbb{R}	Normal	5.6	3
$\lambda_{c,12}$	\mathbb{R}	Normal	0	50
$\lambda_{d,11}$	\mathbb{R}	Normal	16.81	5
$\lambda_{d,12}$	\mathbb{R}	Normal	0	10^3
$h_{c,11}$	\mathbb{R}	Normal	0	6
$h_{c,22}$	\mathbb{R}	Normal	0	50
$h_{d,11}$	\mathbb{R}	Normal	0	10
$h_{d,22}$	\mathbb{R}	Normal	0	10^4
γ	\mathbb{R}^+	Gamma	10	5
ψ	\mathbb{R}^+	Gamma	1.5	.5
δ	$[0, 1)$	Beta	.995	.003
$\sqrt{\alpha}$	\mathbb{R}^+	Inverse Gamma	.01	4
ϕ_x	\mathbb{R}^+	Gamma	0.05	0.05
ϕ_d	\mathbb{R}^+	Gamma	5	6
$\tilde{\phi}_{11}$	$(-1, 0)$	Uniform	-.9999	-.0001
$\tilde{\phi}_{12}$	$(-1, 1)$	Uniform	-.9999	.9999
$\tilde{\phi}_{22}$	$(-1, 0)$	Uniform	-.9999	-.0001
ρ_{cd}	$[0, 1)$	Beta	.2	.1
σ_2	\mathbb{R}^+	Inverse Gamma	$.25 \times 10^{-4}$	4

Notes: Para (1) and Para (2) list the means and the standard deviations for Beta, Gamma, and Normal distributions; s and ν for the Inverse Gamma distribution, where $p_{IG}(\sigma|\nu, s) \propto \sigma^{-\nu-1} e^{-\nu s^2/2\sigma^2}$, a and b for the Uniform distribution from a to b .

Table 2: POSTERIOR DISTRIBUTION

Parameters	Quadratic			Linear	
	Prior 90%	Posterior Mode	Posterior 90%	Posterior Mode	Posterior 90%
$\lambda_{c,11}$	[0.71, 10.75]	5.94	[-1.48, 8.27]	2.65	[-0.57, 5.33]
$\lambda_{c,12}$	[-85.72, 79.41]	27.83	[-68.25, 49.33]	58.12	[-89.35, 100.99]
$\lambda_{d,11}$	[8.54, 24.99]	17.69	[15.77, 29.07]	24.81	[17.28, 30.99]
$\lambda_{d,12}$	[-1539, 1796]	96.84	[-278.66, 411.92]	663.87	[-802.78, 1462.41]
$h_{c,11}$	[-10.40, 9.07]	-11.01	[-12.08, 3.97]	0	Fixed
$h_{c,22}$	[-80.37, 82.86]	0.01	[-2.49, 0.82]	0	Fixed
$h_{d,11}$	[-15.96, 16.75]	9.69	[-8.71, 21.46]	0	Fixed
$h_{d,22}$	[-1615, 1674]	1310.48	[-1453, 1512]	0	Fixed
γ	[2.15, 17.66]	7.46	[0.95, 7.26]	3.63	[2.37, 19.05]
ψ	[0.61, 1.69]	1.36	[1.06, 1.90]	1.17	[0.75, 1.95]
δ	[0.989, 0.999]	0.997	[0.989, 0.998]	0.996	[0.989, 0.999]
$\sqrt{\alpha}$	[0.006, 0.022]	0.024	[0.023, 0.039]	0.011	[0.008, 0.050]
ϕ_x	[0.00001, 0.1219]	0.397	[0.203, 0.466]	0.280	[0.200, 0.408]
ϕ_d	$[7 \times 10^{-7}, 12.5]$	3.42	[2.83, 3.96]	3.65	[2.53, 4.25]
$\tilde{\phi}_{11}$	[-0.978, -0.082]	-0.014	[-0.063, -0.0002]	-0.002	[-0.232, -0.0001]
$\tilde{\phi}_{12}$	[-0.994, 0.799]	-0.869	[-0.991, 0.488]	-0.171	[-0.977, 0.782]
$\tilde{\phi}_{22}$	[-0.990, -0.102]	-0.005	[-0.088, -0.003]	-0.015	[-0.086, -0.003]
ρ_{cd}	[0.043, 0.348]	0.218	[0.198, 0.480]	0.388	[0.196, 0.501]
σ_2	$[1 \times 10^{-5}, 4.5 \times 10^{-5}]$	0.0001	[0.00006, 0.00014]	0.0001	[0.00007, 0.00015]

Notes: If we fix $h_{c,11}, h_{c,22}, h_{d,11}, h_{d,22}$ at 0, we get the linear model as a special case.

Table 3: POSTERIOR PREDICTIVE MOMENTS

Moments	Data	Quadratic			Linear			Linear with BY (2004)		
		5%	50%	95%	5%	50%	95%	5%	50%	95%
Mean (Δc)	2.01	0.23	1.48	2.93	0.82	1.63	2.03	-0.04	1.43	1.77
StdDev (Δc)	2.96	2.86	3.64	5.08	2.86	3.58	4.43	2.91	3.87	5.35
AC1 (Δc)	0.31	-0.11	-0.07	0.10	-0.11	-0.07	0.02	-0.11	-0.07	0.004
Mean (Δd)	1.30	-13.24	0.18	5.01	-2.49	2.76	7.55	-7.35	-2.00	1.64
StdDev (Δd)	10.94	10.41	13.02	17.51	10.78	13.42	16.78	10.67	14.98	20.23
AC1 (Δd)	0.17	0.05	0.10	0.27	0.04	0.15	0.44	0.11	0.19	0.42
Corr ($\Delta c, \Delta d$)	0.51	0.11	0.29	0.43	0.14	0.30	0.45	0.18	0.31	0.45
Mean (r_d)	8.80	-9.28	2.57	7.81	1.48	6.55	10.36	-0.89	2.73	5.76
StdDev (r_d)	18.40	24.49	30.94	39.22	22.12	28.43	34.69	23.97	33.56	44.95
AC1 (r_d)	0.07	-0.08	-0.05	-0.02	-0.09	-0.05	-0.002	-0.05	-0.03	-0.001
Corr ($\Delta c, r_d$)	0.13	-0.07	0.002	0.08	-0.07	0.02	0.11	-0.06	-0.003	0.07
Mean (r_f)	1.47	-0.45	0.76	2.73	-0.11	0.75	1.55	-0.66	0.94	1.56
StdDev (r_f)	3.11	3.10	3.16	3.48	3.10	3.15	3.43	3.12	3.18	3.34
AC1 (r_f)	0.82	0.04	0.07	0.19	0.04	0.06	0.18	0.05	0.08	0.15
Mean (z_d)	3.36	2.22	3.02	7.59	2.53	3.17	3.95	1.60	2.58	3.09
StdDev (z_d)	0.46	0.55	0.75	1.10	0.33	0.68	0.85	0.46	0.70	1.04
AC1 (z_d)	0.90	0.85	0.92	0.93	0.71	0.92	0.94	0.84	0.89	0.93

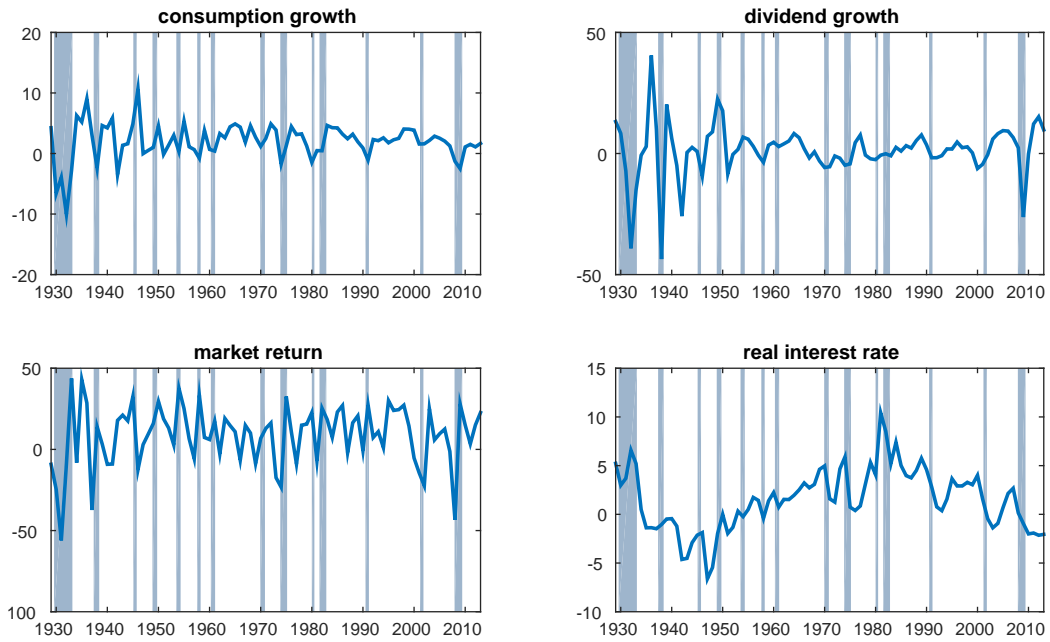
Notes: If we fix $h_{c,11}, h_{c,22}, h_{d,11}, h_{d,22}$ at 0, we get the linear model as a special case.

Table 4: PREDICTABILITY OF CASH FLOWS AND EXCESS MARKET RETURN BY THE PRICE-DIVIDEND RATIO

Moments	Data	Quadratic		Linear		Linear with BY (2004)	
		50%	90% Interval	50%	90% Interval	50%	90% Interval
$R^2(\Delta c_{t+1})$	0.075	0.006	[0,0.165]	0.004	[0, 0.094]	0.018	[0.003, 0.075]
$R^2(\sum_{j=1}^5 \Delta c_{t+j})$	0.001	0.056	[0.0006,0.452]	0.03	[0.0003, 0.337]	0.180	[0.078, 0.382]
$R^2(\Delta d_{t+1})$	0.08	0.1	[0.044, 0.307]	0.145	[0.029, 0.587]	0.211	[0.121, 0.492]
$R^2(\sum_{j=1}^5 \Delta d_{t+j})$	0.02	0.196	[0.073, 0.458]	0.267	[0.049, 0.512]	0.358	[0.236, 0.551]
$R^2(r_{d,t+1} - r_{f,t})$	0.02	0.027	[0.003,0.051]	0.011	[0.0001, 0.035]	0.007	[0.0001, 0.012]
$R^2(\sum_{j=1}^5 r_{d,t+j} - r_{f,t+j-1})$	0.27	0.164	[0.021,0.266]	0.088	[0.001, 0.178]	0.06	[0.003, 0.098]

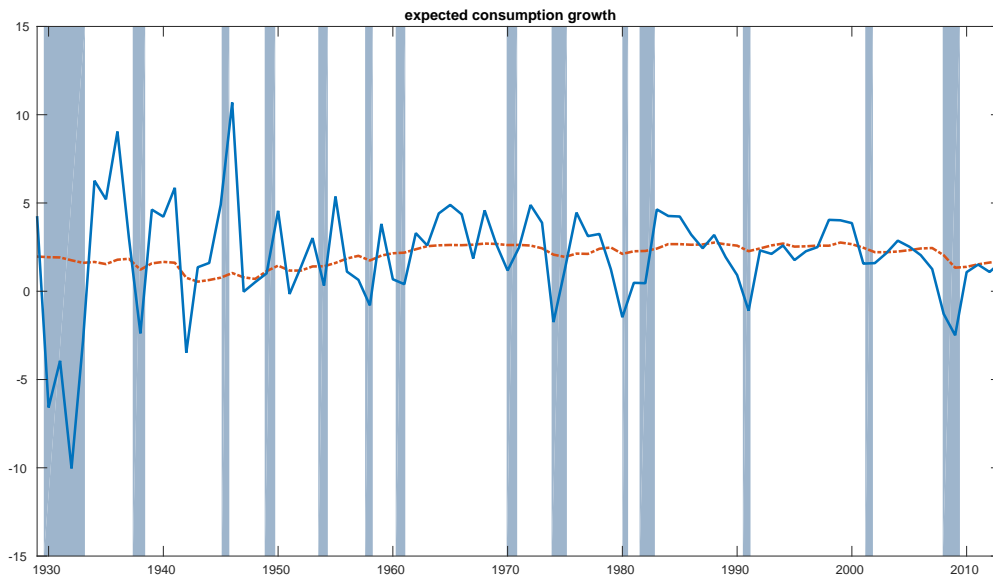
Notes: If we impose restrictions on market price of risks and state variable dynamics under the risk-neutral measure in the linear model, we get BY (2004) specification.

Figure 1: ANNUAL DATA FOR CASH FLOW AND ASSET PRICES: 1890-2013



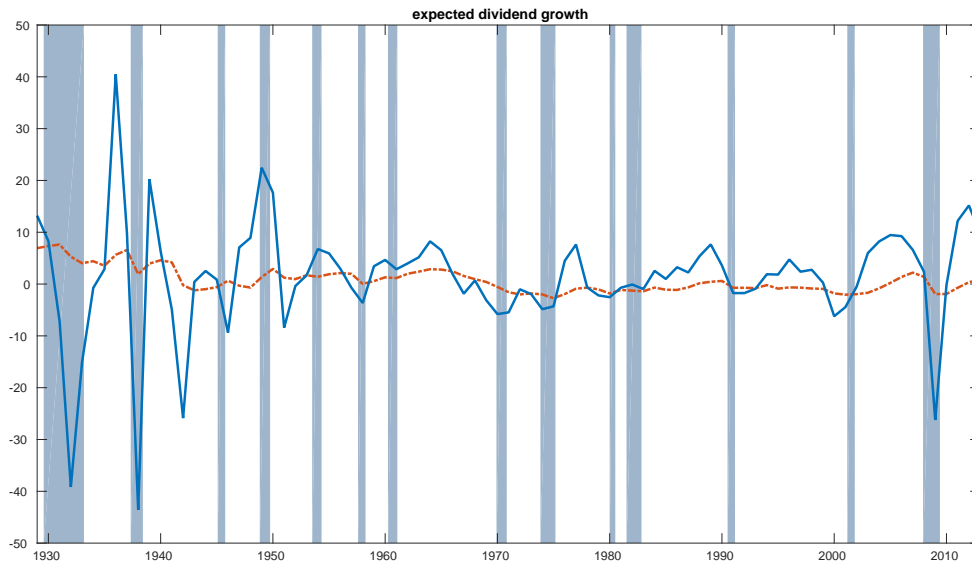
Notes: The colored area denotes the recession period identified by the National Bureau of Economic Research.

Figure 2: EXPECTED CONSUMPTION GROWTH



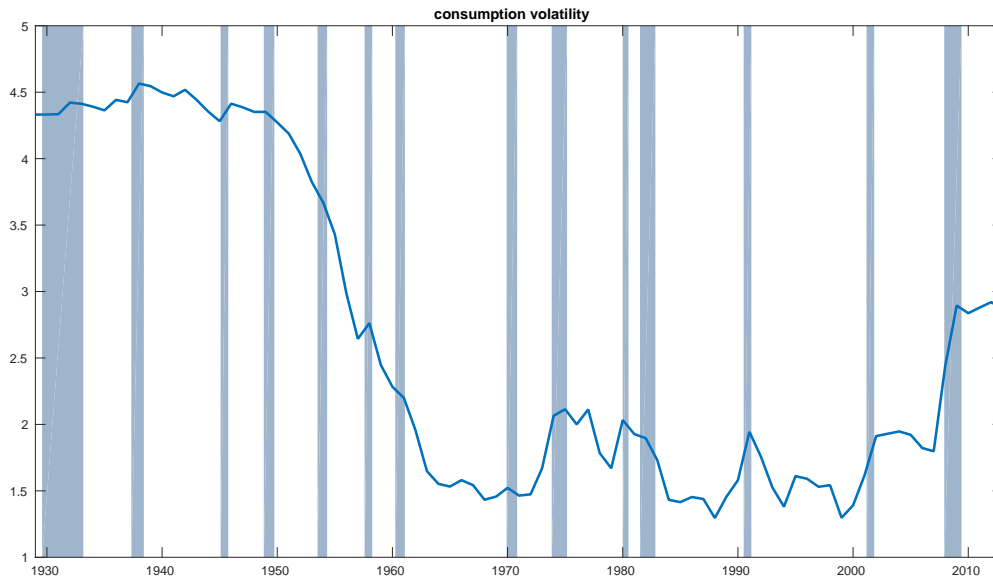
*Notes:*The colored area denotes the recession period identified by the National Bureau of Economic Research. The solid line denotes realized consumption growth and the dash-dot line represents expected consumption growth implied by the quadratic model at the posterior mode.

Figure 3: EXPECTED DIVIDEND GROWTH



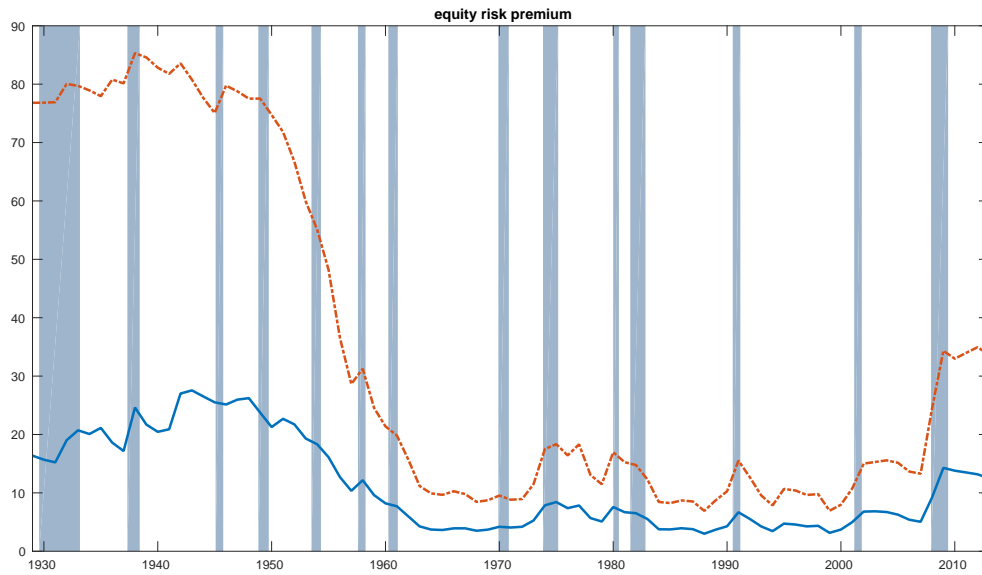
Notes: The colored area denotes the recession period identified by the National Bureau of Economic Research. The solid line denotes realized dividend growth and the dash-dot line represents expected dividend growth implied by the quadratic model at the posterior mode.

Figure 4: CONSUMPTION VOLATILITY



*Notes:*The colored area denotes the recession period identified by the National Bureau of Economic Research. Estimates of consumption volatility are obtained from the posterior mode of the quadratic model.

Figure 5: EQUITY RISK PREMIUM



Notes: The colored area denotes the recession period identified by the National Bureau of Economic Research. The solid line denotes the model-implied equity risk premium at the posterior mode of the quadratic model. The dash-dot line represents the model-implied equity risk premium at the posterior mode of the quadratic model when the market price of risk is restricted to be constant by setting $h_{c,11}$, $h_{c,22}$, $h_{d,11}$ and $h_{d,22}$ at zero.